Line Emission

ASTR 5420 – Stellar Evolution & Interiors

Thursday, March 6, 2014

1 Opacity Sources

- electron scattering (Thomson)
- free-free absorption
- bound-free absorption (photo-ionization)
- H^- opacity
- Bound-bound absorption (atomic & molecular lines)
- scattering on molecules and grains

General form:

$$\chi \propto \rho^n T^{-s} \tag{1}$$

This form is called Kramer's Law.

1.1 Electron scattering

cross-section:

$$\sigma = \frac{\text{number of events per time per target}}{\text{incident photon flux}}\,\text{cm}^2$$

mean free path

$$\ell = \frac{1}{\sigma n_e}\,[\text{cm}]$$

scattering opacity:

$$\kappa = \frac{\sigma n_e}{\rho} = \frac{\sigma}{\bar{\mu}_e}$$

Thomson scattering:

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 0.6652 \times 10^{-24}\,\text{cm}^2$$

where $e^2/m_e c^2$ is the classical electron radius. Then

$$\kappa_e = 0.2(1+X)\ \text{cm}^2\ \text{g}^-1$$

$s = n = 0$. This opacity depends on Saha equation and ionization states, as other opacities will also show to be.

1.2 Free-free absorption

Reverse of Bremsstrahlung emission.

Bremsstrahlung emission occurs when an electron is accelerated in the electric potential of an ion. (Ion has charge Z_c, electron has charge e-, and the impact parameter is b. The energy lost by the electron is emitted as a photon. The ion absorbs the angular momentum difference.

The reverse process is free-free absorption.

$$\kappa_{\rm ff} \approx 10^{23} \frac{Z_c^2}{\mu_e \mu_I} \left(\frac{\rho}{\rm g\ cm^{-3}} \right) \left(\frac{T}{\rm K} \right)^{-3.5} \ {\rm cm^2\ g^{-1}} \tag{2}$$

where Z_C is the average nuclear charge. Or

$$\kappa_{\rm ff} \approx 3.8 \times 10^{22} (1+X)(X+Y+B)\rho T^{-7/2} \tag{3}$$

where

$$B = \sum_i \frac{X_i Z_i^2}{A_i}$$

Aside: Text refers to opacity laws in terms of $\kappa = \kappa_0 \rho^n T^{-s}$. When $n = 1$ and $s = 3.5$, this is refered to as a Kramers' opacity.

Requires free electrons. When no free electrons, $\mu_e \to \infty$

1.3 Bound-free absorption

This results from excitation of electrons bound to atoms. If they absorb enough energy, they are ionized.

Bound-bound absorption creates discrete lines.

Bound-free = photoionization.

Molecules may be photodissociated.

$$\kappa_{\rm bf} \approx 4 \times 10^{25} Z(1+X) \left(\frac{\rho}{\rm g\ cm^{-3}} \right) \left(\frac{T}{\rm K} \right)^{-3.5} \ {\rm cm^2\ g^{-1}} \tag{4}$$

1.4 Bound-bound absorption

Atomic and molecular lines – see section on Line Broadening below.

1.5 H^- opacity

Important in atmospheres of cooler stars. Hotter stars, H becomes ionized. Need H in neutral form, plus lots of free electrons from metals, especially alkali metals like Na, K, Ca, Al.

$$\kappa_{\rm H^-} \approx 2.5 \times 10^{-31} \left(\frac{Z}{0.02} \right) \left(\frac{\rho}{\rm g\ cm^{-3}} \right)^{1/2} \left(\frac{T}{K} \right)^{9} \ {\rm cm^2\ g^{-1}} \tag{5}$$

1.6 Conductive opacities

$$\frac{1}{\kappa} = \frac{1}{\kappa_{\rm rad}} + \frac{1}{\kappa_{\rm cond}}$$

1.7 Molecules

Molecular lines: lots of them. Typically only important in cool stars, because hot stars destroy molecules.

Rayleigh scattering: $\sigma \propto \lambda^{-4}$. Responsible for blue skies, red sunrises and sunsets, and interstellar reddening.

2 Line Broadening

(references: LeBlanc §4.3, HKT 4.8)

2.1 Natural line broadening

is caused by Heisenberg uncertainty principle:

$$\Delta E \Delta t \geq \frac{h}{4\pi}$$

In other words, the energy of the transition has a certain width, $\Delta E = h\Delta\nu$. This gives the Lorentz profile:

$$\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}$$

where $\gamma/2\pi$ is the full width at half-maximum, and

$$\gamma = \sum_{n'<n} A_{nn'} \tag{6}$$

where the summation is carried out over all states n' of lower energy.

The lifetime of the Lyα transition is $\tau = 1.6 \times 10^{-9}$ s, and $\gamma = 1/\tau$. So $\tau = \ldots$?

Recall, the absorption coefficient, omitting stimulated emission, is given by

$$\alpha_\nu = \frac{h\nu}{4\pi}\phi(\nu) n_i B_{ij} = \frac{\pi e^2}{m_e c} f_{ij}\phi(\nu) n_i = \kappa\rho$$

where f_{12} is the oscillator strength:

$$B_{ij} = \frac{4\pi^2 e^2}{h\nu_{ij} m_e c} f_{ij}$$

It is a quantum mechanical correction to the classical value.

Differs by a factor of n_i from definition of α in LeBlanc ($k = \kappa$) Recall that the cross section is related to the opacity by

$$\kappa = \sigma n_i/\rho$$

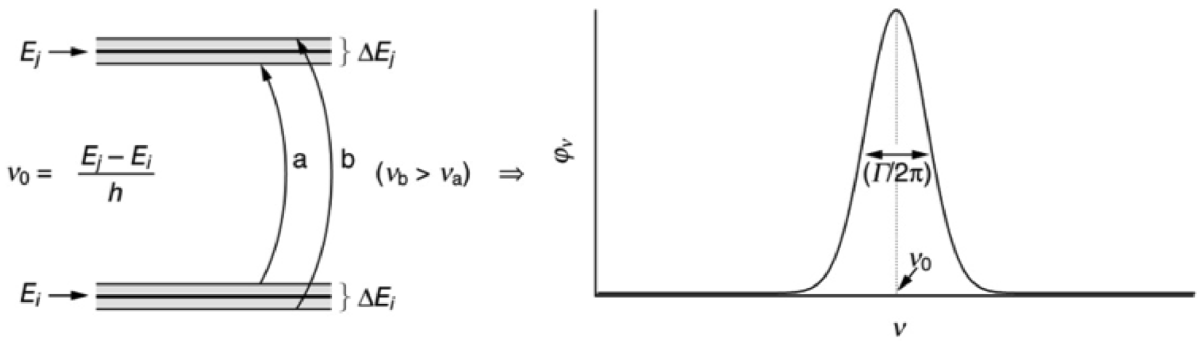

Figure 4.4 Illustration of the effect of the width of atomic energy levels due to the uncertainty principle on the profile of an atomic transition between levels i and j. Here, the energy levels E_i and E_j are those obtained by Schrödinger's equation, while ΔE_i and ΔE_j are the corresponding uncertainties predicted by Heisenberg's uncertainty principle. The value $\Gamma/2\pi$ represents the full width of the profile at half-intensity.

and you get

$$\sigma(\text{Lorentz}) = \frac{e^2}{mc} f_{ij} \frac{(\gamma/4\pi)}{(\nu-\nu_0)^2 - (\gamma/4\pi)^2} \tag{7}$$

2.2 Doppler Broadening

Thermal motions of the gas means that the atoms move at different velocities along the line of sight, according to the Boltzmann distribution.

The Doppler shift:

$$\nu' = \nu\left(1 - \frac{v}{c}\mu\right)$$

where μ is the cosine of the angle with respect to the observer.

Velocity distribution: recall (HKT 3.3) that the momentum distribution of an ideal gas is as below, we want to reduce it to one line of sight.

$$\begin{aligned}
\frac{dn(\mathbf{p})}{n} &= \frac{1}{(2\pi mkT)^{3/2}} \exp[-p^2/2mkT]\, d^3\mathbf{p} \\
\frac{dn(\mathbf{p})}{n} &= \frac{1}{(2\pi mkT)^{3/2}} \exp[-p^2/2mkT]\, d^3\mathbf{p} \\
\frac{dn(p_x,p_y,p_z)}{n} &= \frac{1}{(2\pi mkT)^{3/2}} \exp\left[\frac{-p_x^2 - p_y^2 - p_z^2}{2mkT}\right] dp_x dp_y dp_z \\
\frac{dn(p_x)}{n} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(2\pi mkT)^{3/2}} \exp\left[\frac{-p_x^2 - p_y^2 - p_z^2}{2mkT}\right] dp_x dp_y dp_z \\
\frac{dn(p_x)}{n} &= \frac{1}{(2\pi mkT)^{1/2}} \exp\left(\frac{-p_x^2}{2mkT}\right) dp_x \\
\frac{dn(v_x)}{n} &= \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(\frac{-mv_x^2}{2kT}\right) dv_x
\end{aligned}$$

The Doppler-shifted line profile looks like a Gaussian.
The Doppler broadening width is the Gaussian width:

$$\Delta v = \sqrt{\frac{2kT}{m}}$$

$$\Delta\nu_D = \nu_0 \frac{\Delta v}{c} = \nu_0 \frac{1}{c}\left(\frac{2kT}{m}\right)^{1/2}$$

To get a Doppler broadened Lorentz profile, we need to convolve the two together:

$$\begin{aligned}
\kappa(\nu) &= \int \kappa\left(\nu - \nu\frac{v}{c}\right)\frac{dn(v)}{n} \\
&= \int_{-\infty}^{\infty} \frac{1}{\bar{\mu}}\frac{\pi e^2}{m_e c} f_{ij}\phi(\nu)\left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(\frac{-mv_x^2}{2kT}\right) dv \\
&= \frac{1}{\bar{\mu}}\frac{\pi e^2}{m_e c} f_{ij}\int_{-\infty}^{\infty}\frac{1}{\pi^{1/2}}\frac{1}{\Delta v}\frac{\frac{\Gamma}{4\pi^2}\exp\left(\frac{-v_x^2}{\Delta v^2}\right)}{(\nu - \nu v/c - \nu_0)^2 + (\Gamma/4\pi)^2} dv \\
&= \frac{1}{\bar{\mu}}\frac{e^2}{m_e c} f_{ij}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{\frac{\Gamma}{4\pi}\exp\left(\frac{-v_x^2}{\Delta v^2}\right)}{(\nu - \nu v/c - \nu_0)^2 + (\Gamma/4\pi)^2}\frac{dv}{\Delta v} \\
&= \frac{1}{\bar{\mu}}\frac{e^2}{m_e c} f_{ij}\phi(\nu)
\end{aligned}$$

where $\phi(\nu)$ is the Voigt function. We could also integrate over frequency, with $v/c = \nu - \nu_0$

$$\phi(\nu) = \int_{-\infty}^{\infty}\frac{\Gamma/4\pi^2}{(\nu-\nu')^2 + (\Gamma/4\pi)^2}\frac{e^{-(\nu'-\nu_0)^2/(\Delta\nu_D)^2}}{\Delta\nu_D\sqrt{\pi}} d\nu' \tag{8}$$

3 Other broadening mechanisms

3.1 Stellar Rotation

Width of the line is given by $v \sin i$.

3.2 Pressure or Collisional Broadening

Collisional broadening follows a Lorentz profile.

$$\gamma \to \Gamma = \gamma + 2\nu_{\mathrm{col}}$$

where ν_{col} is the frequency of collisions.

$$\frac{1}{\nu_{\mathrm{col}}} = \Delta t = \frac{\ell}{v}$$

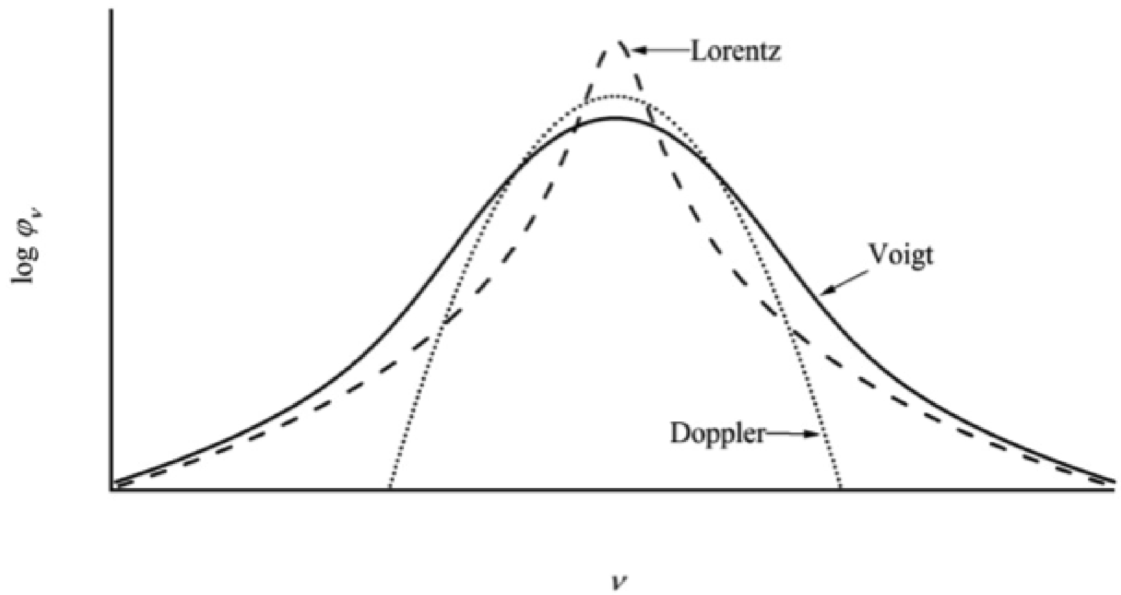

Figure 4.6 Lorentz, Voigt and Doppler profiles of a hypothetical atomic line.

`LeBlanc Figure 4.10 (p. 129)`

where ℓ is the mean free path, and v is the speed.

$$\ell = \frac{1}{n\sigma}$$

The speed can be estimated from the average kinetic energy of a molecule in a gas, $u = kT$, so $\frac{1}{2}m_H v^2 = kT$ so

$$v = \sqrt{\frac{2kT}{m_H}}.$$

For different molecules, cross sections are measured. For hydrogen, collisional cross-section is

$$\sigma_H \approx 3.53 \times 10^{-16}\ \mathrm{cm}^2.$$

What is Γ for the photosphere of the sun?
(Need to get density from

$$P(\tau_p) = \frac{2g_s}{3\kappa_p} \tag{9}$$

$$\frac{aT_{\mathrm{eff}}^4}{3} + \frac{\rho_p k T_{\mathrm{eff}}}{\bar{\mu}} = \frac{2}{3}\frac{g_s}{\kappa} \tag{10}$$

assuming radiation pressure is small.)

Surface gravity can be measured be examining pressure broadening.

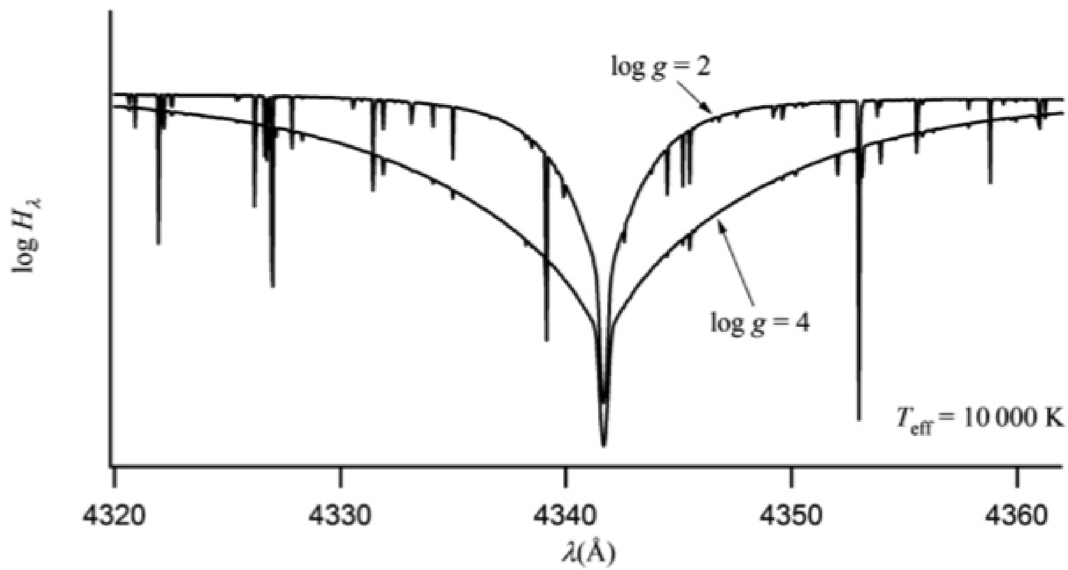

Figure 4.11 The surface flux within the H_γ line at the surface of atmospheres with $T_{\rm eff} = 10\,000$ K but with different surface gravities typical of main-sequence ($\log g = 4$) and supergiant ($\log g = 2$) stars. Other atomic lines from various metals are also seen within the H_γ line.

4 Equivalent Width

(reference: LeBlanc 4.4)

$$\mathrm{EW} = \int \frac{(F_c - F_\lambda)\, d\lambda}{F_c}$$

units of Å, or wavelength.

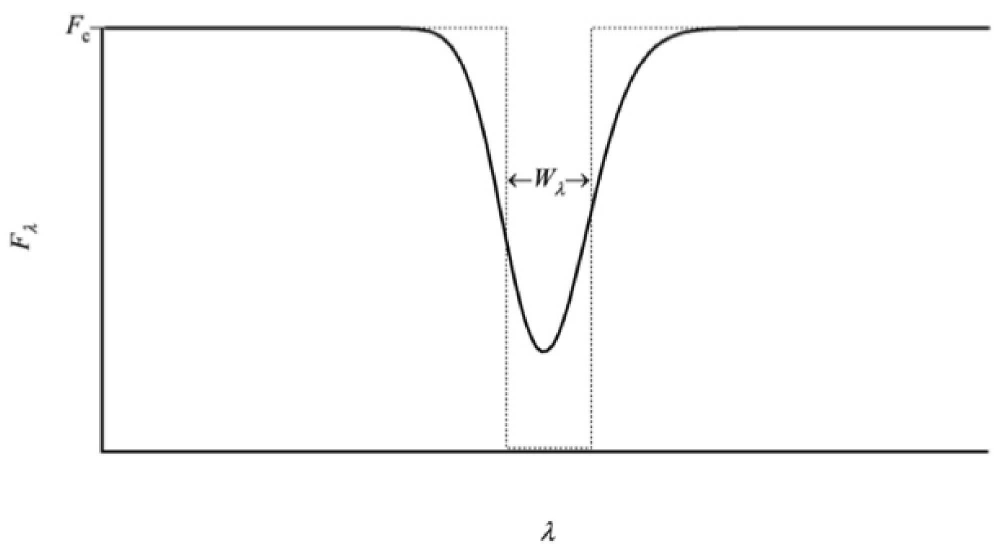

Figure 4.14 Schematic definition of the equivalent width (W_λ) of an atomic line. The fictitious rectangular line which absorbs all photons within it has a width such that it absorbs the same quantity of energy as the atomic line to which it is associated. The quantity F_c is the flux of the continuum.

5 Curve of Growth

(references: LeBlanc §4.4.3, HKT 4.8)

5.1 Weak lines

We generally see to an optical depth of $\tau \approx 2/3$ into the stellar surface. Since the opacity is higher in the line, we see to a shallower depth.

- optical depth in continuum: $d\tau_c = -\chi_c \rho \, dr$
- optical depth in line: $d\tau_\lambda = -\chi_\lambda \rho \, dr$
- opacities: $\chi_\lambda = \chi_c + \chi_1$ where χ_1 is opacity in just the line

$$\frac{F_c - F_\lambda}{F_c} \approx \frac{B_\lambda(\tau_c = 2/3) - B_\lambda(\tau_\lambda = 2/3)}{B_\lambda(\tau_c = 2/3)}$$

Let the geometric depth at which $\tau_\lambda = 2/3$ be where $\tau_c = 2/3 + \Delta\tau$, $\Delta\tau < 0$.

$$\begin{aligned}
\frac{\tau_c}{\tau_\lambda} &= \frac{\chi_c}{\chi_\lambda} \\
\frac{2/3 + \Delta\tau}{2/3} &= \frac{\chi_c}{\chi_c + \chi_1} \\
\Delta\tau &= \frac{2\chi_c}{3(\chi_c + \chi_1)} - \frac{2}{3} \\
\Delta\tau &= -\frac{2\chi_1}{3(\chi_c + \chi_1)} \approx -\frac{2\chi_1}{3\chi_c}
\end{aligned}$$

for $\chi_1 \ll \chi_c$: weak line.

$$\begin{aligned}
B_\lambda(\tau_\lambda = 2/3) &= B_\lambda(\tau_c = 2/3 + \Delta\tau) \approx B_\lambda(\tau_c = 2/3) + \Delta\tau \frac{dB_\lambda}{d\tau_c}(\tau_c = 2/3) \\
&= B_\lambda(\tau_c = 2/3) - \frac{2\chi_1}{3\chi_c}\frac{dB_\lambda}{d\tau_c}
\end{aligned}$$

$$\frac{F_c - F_\lambda}{F_c} \approx \frac{2}{3}\frac{\chi_1}{\chi_c}\frac{d\ln B_\lambda}{d\tau_c}(\tau_c = 2/3)$$

roughly proportional to χ_1.

5.2 Curve of Growth

Line depth increases roughly linearly at first. Bottoms out at the flux at the stellar surface ($\tau = 0$). At this point the line is **saturated**, and line depth is no longer linear.

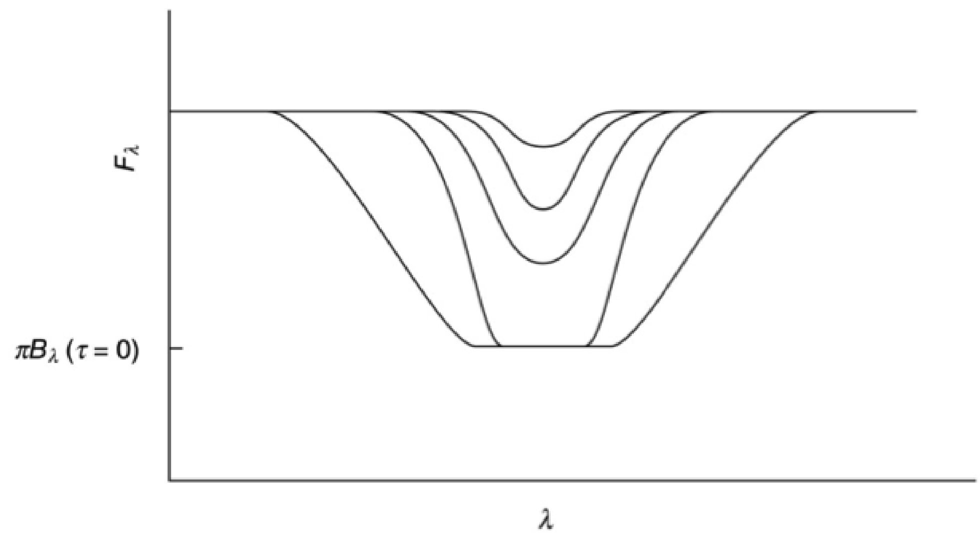

Figure 4.17 Illustration of the varying shape of an atomic line as the abundance increases. It goes from an unsaturated to a saturated condition. As the line deepens, the flux eventually attains a minimum, which when assuming LTE is equal to $\pi B_\lambda(\tau = 0)$.

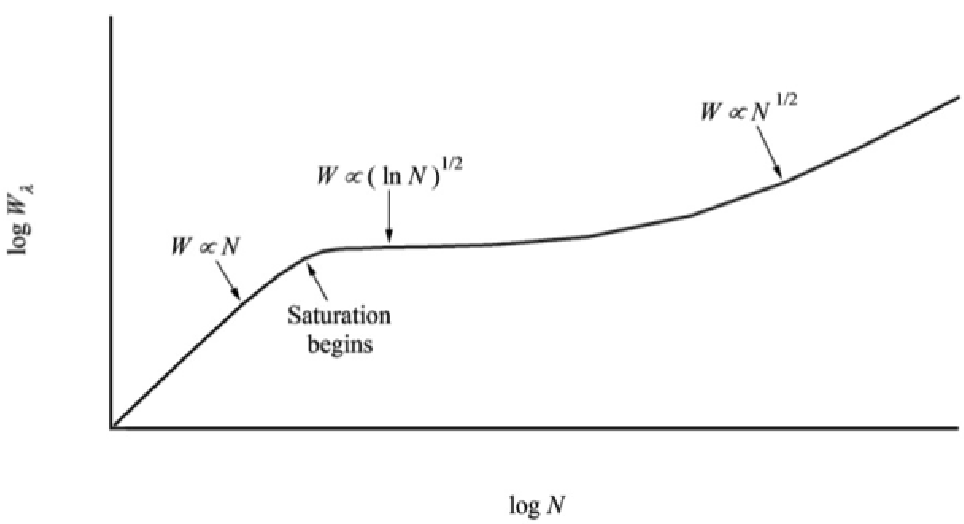

Figure 4.18 Illustration of the equivalent width (W_λ) as a function of the abundance (N) of the species for a given atomic line (commonly called the curve of growth). The dependence of the equivalent width with respect to abundance for the various parts of the curve is given in the figure. The approximate position where the line begins to be saturated is also shown in this figure.

```
------ tau=0         ------- -------- F(tau=2/3)
                            | |
                            |_|_______ F(tau_line=2/3)

------ tau_line=2/3

------ tau=2/3       ---------------- F(tau=0)
```

Line saturates when opacity is so high, you only see the temperature at $\tau = 0$. Cannot calculate good abundances for species whose lines are saturated.

6 Spectral Types

(reference: HKT 4.7)

7 Putting things in context

A stellar interior model gives a theoretical T_{eff} and Luminosity of a star. But the atmosphere is a very small mass fraction of the star – the model doesn't necessarily tell you what the atmospheric conditions are like.

Stellar atmosphere model answers the questiong of how does the star actually emit. The inputs needed for such models include luminosity, surface gravity, and chemical composition. A typical stellar atmosphere model doesn't actually depend much on the details of the interior model.

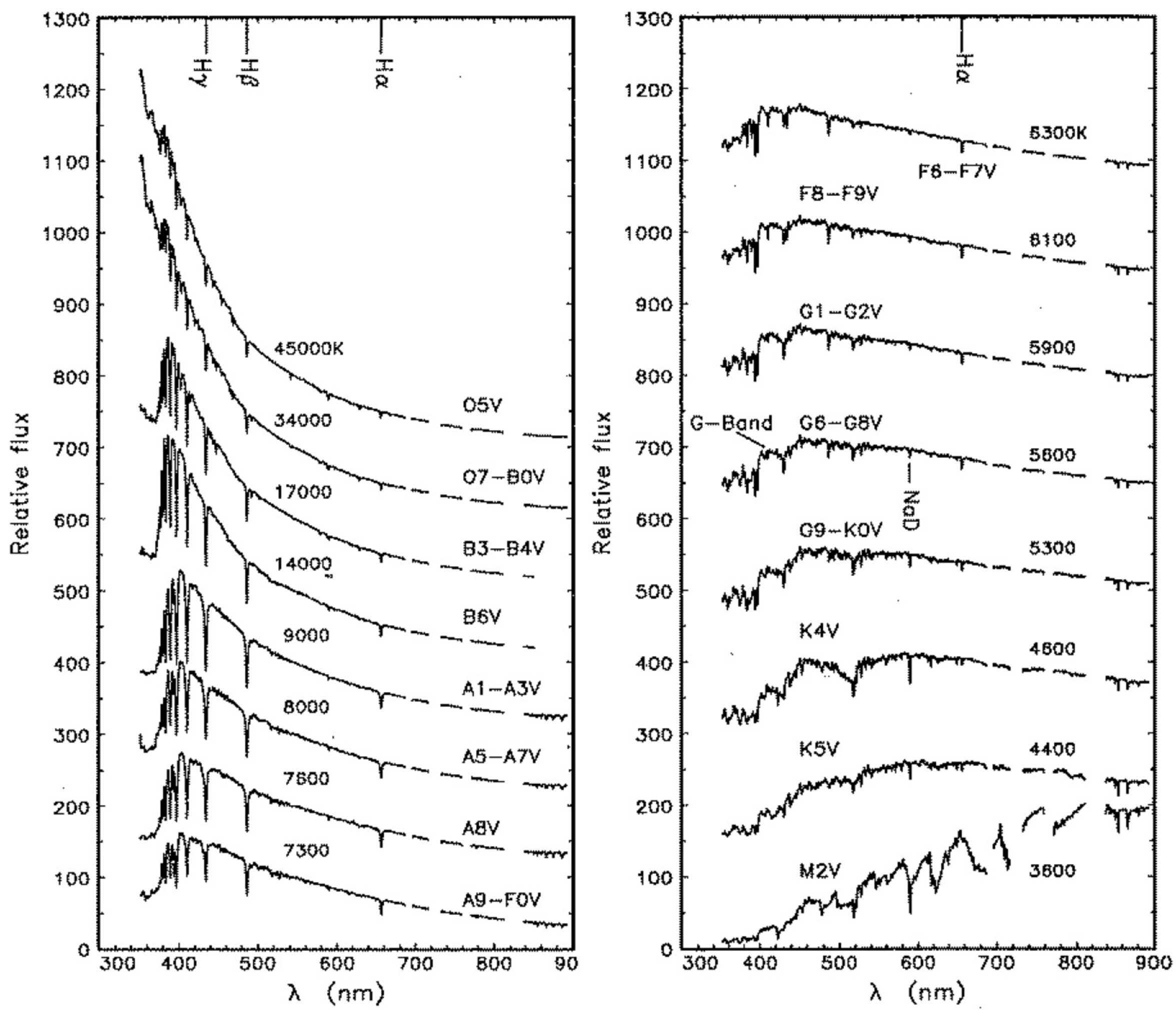

Fig. 4.8. Spectra of main sequence (luminosity class V) stars of spectral classes O–M derived from the work of Silva and Cornell (1992).

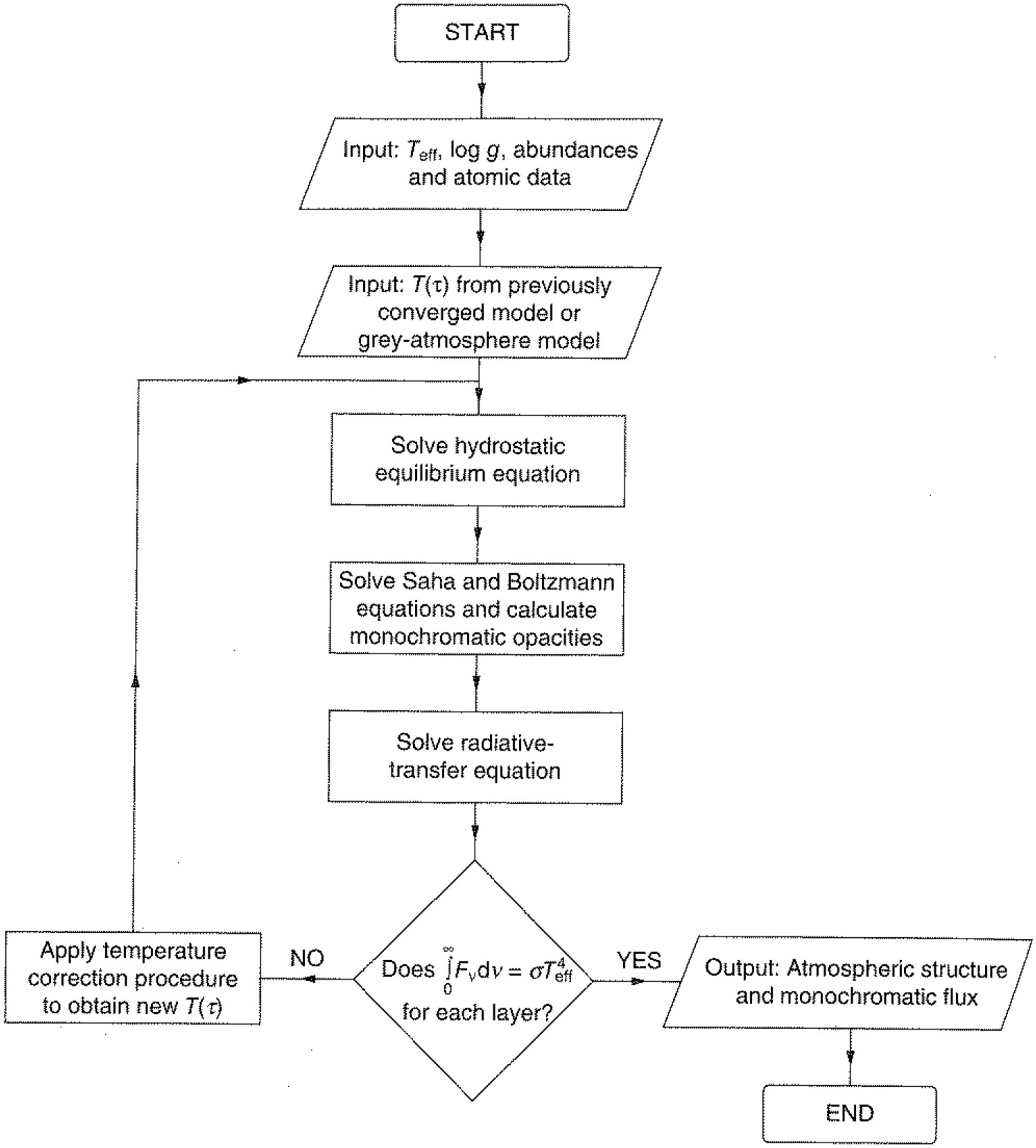

Figure 4.19 Flowchart of the algorithm used for atmospheric modelling of a plane-parallel model. Once the radiative-transfer equation is solved, radiative pressure that is not mentioned in this figure or in the text may also be included in the hydrostatic equilibrium equation for subsequent iterations (see optional Section 3.12). Figure reproduced and adapted with permission from François Wesemaël.