about the expansion of the universe and is independent of time as the universe expands or contracts.

Unfortunately, computing the numerical value of ε_{vac} is an exercise in quantum field theory that has not yet been successfully completed. It has been suggested that the natural value for the vacuum energy density is the Planck energy density,

$$\varepsilon_{vac} \sim \frac{E_P}{\ell_P^3} \qquad (???). \tag{4.75}$$

As we've seen in Chapter 1, the Planck energy is large by particle physics standards ($E_P = 1.22 \times 10^{28}\,\text{eV} = 540$ kilowatt-hours), while the Planck length is small by anybody's standards ($\ell_P = 1.62 \times 10^{-35}\,\text{m}$). This gives an energy density

$$\varepsilon_{vac} \sim 3 \times 10^{132}\,\text{eV m}^{-3} \qquad (!\,!\,!). \tag{4.76}$$

This is 123 orders of magnitude larger than the current critical density for our universe, and represents a spectacularly bad match between theory and observations. Obviously, we don't know much yet about the energy density of the vacuum! This is a situation where astronomers can help particle physicists, by deducing the value of ε_Λ from observations of the expansion of the universe. By looking at the universe at extremely large scales, we are indirectly examining the structure of the vacuum on extremely small scales.

Exercises

4.1 Suppose the energy density of the cosmological constant is equal to the present critical density $\varepsilon_\Lambda = \varepsilon_{c,0} = 4870\,\text{MeV m}^{-3}$. What is the total energy of the cosmological constant within a sphere 1 AU in radius? What is the rest energy of the Sun ($E_\odot = M_\odot c^2$)? Comparing these two numbers, do you expect the cosmological constant to have a significant effect on the motion of planets within the solar system?

4.2 Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.

4.3 If $\rho = 2.7 \times 10^{-27}\,\text{kg m}^{-3}$, what is the radius of curvature R_0 of Einstein's static universe? How long would it take a photon to circumnavigate such a universe?

4.4 Suppose that the universe were full of regulation baseballs, each of mass $m_{bb} = 0.145\,\text{kg}$ and radius $r_{bb} = 0.0369\,\text{m}$. If the baseballs were distributed uniformly throughout the universe, what number density of baseballs would be required to make the density equal to the critical density? (Assume nonrelativistic baseballs.) Given this density of baseballs, how far would you be able to see, on average, before your line of sight intersected a baseball? In fact, we can see galaxies at a distance $\sim c/H_0 \sim 4000\,\text{Mpc}$; does the transparency of the universe on this length scale place useful limits on the number density of intergalactic baseballs? (Note to readers outside North America or Japan: feel free to substitute regulation cricket balls, with $m_{\text{cr}} = 0.160\,\text{kg}$ and $r_{\text{cr}} = 0.0360\,\text{m}$.)

4.5 The principle of wave-particle duality tells us that a particle with momentum p has an associated de Broglie wavelength of $\lambda = h/p$; this wavelength increases as $\lambda \propto a$ as the universe expands. The total energy density of a gas of particles can be written as $\varepsilon = nE$, where n is the number density of particles, and E is the energy per particle. For simplicity, let's assume that all the gas particles have the same mass m and momentum p. The energy per particle is then simply

$$E = (m^2c^4 + p^2c^2)^{1/2} = (m^2c^4 + h^2c^2/\lambda^2)^{1/2} . \qquad (4.77)$$

Compute the equation-of-state parameter w for this gas as a function of the scale factor a. Show that $w = 1/3$ in the highly relativistic limit ($a \to 0$, $p \to \infty$) and that $w = 0$ in the highly nonrelativistic limit ($a \to \infty, p \to 0$).

Exercises

5.1 A light source in a flat, single-component universe has a redshift z when observed at a time t_0. Show that the observed redshift z changes at a rate

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2}. \tag{5.116}$$

For what values of w does the observed redshift increase with time?

5.2 Suppose you are in a flat, matter-only universe that has a Hubble constant $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. You observe a galaxy with $z = 1$. How long will you have to keep observing the galaxy to see its redshift change by one part in 10^6? [Hint: use the result from the previous problem.]

5.3 In a positively curved universe containing only matter ($\Omega_0 > 1$, $\kappa = +1$), show that the present age of the universe is given by the formula

$$H_0 t_0 = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}} \cos^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right) - \frac{1}{\Omega_0 - 1}. \tag{5.117}$$

Assuming $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, plot t_0 as a function of Ω_0 in the range $1 \leq \Omega_0 \leq 3$.

5.4 In a negatively curved universe containing only matter ($\Omega_0 < 1$, $\kappa = -1$), show that the present age of the universe is given by the formula

$$H_0 t_0 = \frac{1}{1-\Omega_0} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}} \cosh^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right). \tag{5.118}$$

Assuming $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, plot t_0 as a function of Ω_0 in the range $0 \leq \Omega_0 \leq 1$.

5.5 One speculation in cosmology is that the dark energy may take the form of "phantom energy" with an equation-of-state parameter $w < -1$. Suppose that the universe is spatially flat and contains matter with a density parameter $\Omega_{m,0}$, and phantom energy with a density parameter $\Omega_{p,0} = 1 - \Omega_{m,0}$ and equation-of-state parameter $w_p < -1$. At what scale factor a_{mp} are the energy density of phantom energy and matter equal? Write down the Friedmann equation for this universe in the limit that $a \gg a_{mp}$. Integrate the Friedmann equation to show that the scale factor a goes to infinity at a finite cosmic time $t_{\rm rip}$, given by the relation

$$H_0(t_{\rm rip} - t_0) \approx \frac{2}{3|1+w_p|}(1-\Omega_{m,0})^{-1/2}. \tag{5.119}$$

This fate for the universe is called the "Big Rip." Current observations of our own universe are consistent with $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_{m,0} = 0.3$, and $w_p = -1.1$. If these numbers are correct, how long do we have remaining until the "Big Rip"?

5.6 Suppose you wanted to "pull an Einstein," and create a static universe ($\dot{a} = 0$, $\ddot{a} = 0$) in which the gravitational attraction of matter is exactly balanced by the gravitational repulsion of dark energy with equation-of-state parameter $-1/3 < w_q < -1$ and energy density ε_q. What is the necessary matter density (ε_m) required to produce a static universe, expressed in terms of ε_q and w_q? Will the curvature of this static universe be negative or positive? What will be its radius of curvature, expressed in terms of ε_q and w_q?

5.7 Consider a positively curved universe containing only matter (the "Big Crunch" model discussed in Section 5.4.1). At some time $t_0 > t_{\text{crunch}}/2$, during the contraction phase of this universe, an astronomer named Elbbuh Niwde discovers that nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance. He then measures H_0 and Ω_0, finding $H_0 < 0$ and $\Omega_0 > 1$. Given H_0 and Ω_0, how long a time will elapse between Dr. Niwde's observations at $t = t_0$ and the final Big Crunch at $t = t_{\text{crunch}}$? What is the highest amplitude blueshift that Dr. Niwde is able to observe? What is the lookback time to an object with this blueshift?

5.8 Consider an expanding, positively curved universe containing only a cosmological constant ($\Omega_0 = \Omega_{\Lambda,0} > 1$). Show that such a universe underwent a "Big Bounce" at a scale factor

$$a_{\text{bounce}} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2}, \tag{5.120}$$

and that the scale factor as a function of time is

$$a(t) = a_{\text{bounce}} \cosh[\sqrt{\Omega_0} H_0 (t - t_{\text{bounce}})], \tag{5.121}$$

where t_{bounce} is the time at which the Big Bounce occurred. What is the time $t_0 - t_{\text{bounce}}$ that has elapsed since the Big Bounce, expressed as a function of H_0 and Ω_0?

5.9 A universe is spatially flat, and contains both matter and a cosmological constant. For what value of $\Omega_{m,0}$ is t_0 exactly equal to H_0^{-1}?

5.10 In the Benchmark Model, what is the total mass of all the matter within our horizon? What is the total energy of all the photons within our horizon? How many baryons are within the horizon?