Quantum Mechanics 2010

Graduate level

Problem.

(a) Determine the eigenvalue of the operator $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ on states where the total spin operator has the eigenvalue S.
(b) Find the eigenvalue of the following operator

$V = a_1\sigma_{1z} + a_2\sigma_{2z} + b\vec{\sigma}_1 \cdot \vec{\sigma}_2$,
if the total spin is S.

Optical Properties of Solids

The coupled excitation of the electromagnetic wave to the transverse optical phonon creates polaritons in solids. The dielectric function of a polariton can be expressed as

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{4\pi N' e^2 / m}{\omega_T^2 - \omega^2} \quad \text{(Eq. 1).}$$

1. Explain the different contributions that the two terms in Eq. 1 represent.

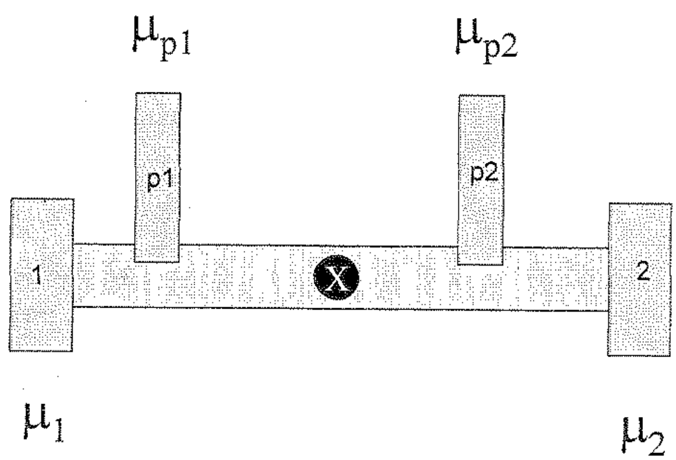
2. ε_0 is defined by $\varepsilon_0 \equiv \varepsilon(\omega = 0)$ while ω_L is defined as the frequency at which the dielectric function vanishes $\varepsilon(\omega_L) \equiv 0$. Using these two definitions and Eq. 1 to prove the Lyddane-Sachs-Teller (LST) relation: $\dfrac{\omega_L^2}{\omega_T^2} = \dfrac{\varepsilon_0}{\varepsilon_\infty}$.

3. Using Eq. 1 and the LST relation to prove that waves do not propagate in the frequency range of $\omega_T < \omega < \omega_L$.

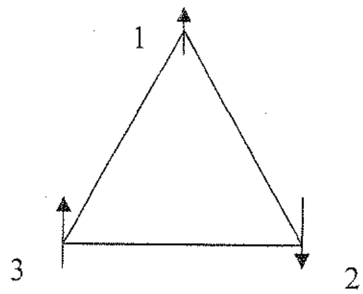
Nanosystems

The four-probe method is usually utilized to measure the resistance of a scatter. The figure shown below is an example of such a measurement setup with contacts 1 and 2, and two additional terminals p1 and p2. X is a scattering center in the conduction channel with a transmission probability of T. Assume that the electrochemical potential in contacts 1 and 2 is μ_1 and μ_2, respectively; also assume that p1 and p2 will measure the local electrochemical potential of either the +k or the −k states ideally.

1. Express $\mu_{p1} - \mu_{p2}$ in term of $\mu_1 - \mu_2$.
2. Derive the channel four-probe resistance R_{4t} measured with p1 and p2.
3. What is the total resistance of the conductor?



Problem: Consider a system of 3 spins S=1/2 arranged in a triangle, as seen in the figure. The spins interact via a (antiferromagnetic) Heisenberg exchange interaction. Focus on the subspace with $S^z=(n_\uparrow-n_\downarrow)/2 = ½$



The Hamiltonian is

$$H = \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1$$

a) Calculate the basis in the |↑>; |↓> representation
b) Calculate the Hamiltonian matrix elements

StatMech:

A 2-state paramagnet has N spins, each with a magnetic moment of μ and each can have energies $+\mu B$ and $-\mu B$ in a magnetic field B. Derive the magnetization M and total energy U of the system as a function of B and temperature T. If the probability for occupying the excited state is half that of the ground state, what is the temperature? If the probability for occupying the excited state is twice that of the ground state, what is the temperature? Based on the relationship $1/T = \partial S/\partial U$, discuss under which circumstance T is negative in this system.

Comprehensive graduate exam January 2010

Phys5870 Experimental Methods of Solid State Physics

Suppose you are using a ^{4}He cryostat. Its base temperature is 4.2 [K].

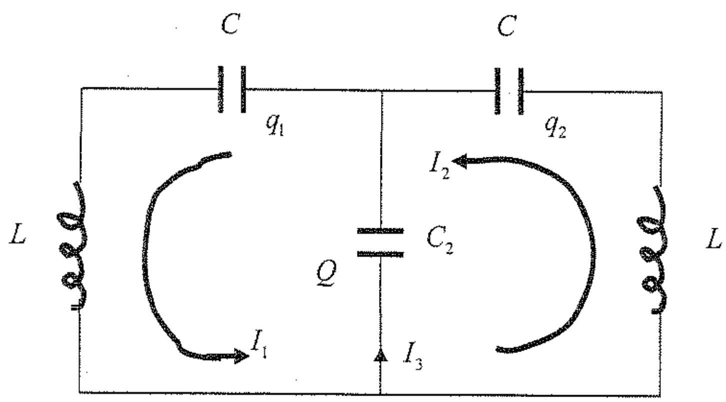
a) Explain how one would obtain a sample temperature of first 1 [K] and later 20 [K].Does one need a different design for the sample chamber and, if so, what are these differences?

b) Which factors do determine how long it takes to achieve thermal equilibrium at 1[K]? Which complications do you expect due to temperature gradients, heat leaks, and the material properties of liquid helium?

c) Now suppose that the cryostat is connected to a liquefier. As you try to cool the cryostat down for the first time and get the liquefier into operation, you observe that the cooling rate at the sample location is extremely low. Assume that the liquefier itself is working well. Describe your strategy of diagnosis, specifically would you add pressure or temperature gauges, and, if so, where would you add them? Which steps you would take to diagnose and remedy the problem and in which sequence would you take these steps?

Comprehensive Exam 2010 Problem
Graduate Physics - Methods of Theoretical Physics

Consider an electrical circuit (inductor, L, and capacitor, C) normal mode charge oscillation analysis (eigenvalue problem) of the charges, q_1, q_2, on the capacitors shown below.



a) Determine the differential equations for the time dependent charges, $q_1(t)$ and $q_2(t)$. **b)** Assume an oscillatory ansatz of frequency, ω, for each time dependent charge, and determine the algebraic (matrix) eigenvalue equation for the initial charge vector, $\mathbf{q}_0 = (q_{10}, q_{20})$. **c)** Determine the normal mode eigenvalue square oscillation frequencies, ω^2. **d)** Determine the orthonormal eigenvector charge amplitudes, $\mathbf{q}_0 = (q_{10}, q_{20})$.

Notes: Consider the Kirchhoff current law (continuity of current into or out of a junction), were the middle current is $I_3 = I_1 - I_2$. Consider the Kirchhoff voltage law (sum of potential drops around a closed loop circuit is zero), where the potential over an inductor is $L\dfrac{dI}{dt}$, the potential over a capacitor is $\dfrac{1}{C}\int I dt$, and the current is $I = \dfrac{dq}{dt}$.

Graduate Physics
Comprehensive examination 2007

Problem 1.

Consider a charge, q, which passes the origin of the laboratory frame, $(x, y, z) = (0,0,0)$, at $t = 0$, and continues to move at a constant velocity, v, in the $\hat{\mathbf{x}}$ direction. Determine the scalar, Φ, and vector, $\mathbf{A}$, potentials in the laboratory frame, expressed in laboratory coordinates, (x, y, z), and time, t.

Hint – consider a Lorentz transformation of the 4-vector potential of the charge from the moving frame boosted back to the laboratory frame.

Problem 2.

Consider an incident particle of mass, m, represented by the time independent 3D Schrödinger equation, where the incident wave function, $\psi_0 e^{i\mathbf{k}\cdot\mathbf{r}}$, is scattered off a potential, $V(\mathbf{r})$. Solve for the scattered wave function, $\psi(\mathbf{r})$, using a Green's function technique: a) written as a 3D spatial integral (which should appear as an integral equation), and b) write down the first order approximate solution (the Born approximation), $\psi_1(\mathbf{r})$, again as an integral.

Hint – the Green's function, $G(\mathbf{r}, \mathbf{r}')$, is of the Helmholtz equation (or 3D harmonic oscillator equation), the Schrödinger equation is $-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$, and the approximate solution is from one iteration using the incident wave function.

Problem 3.

Consider a classical particle, of mass, m, constrained to move on the inside surface of a smooth cone of half angle α, where the gravitational field, g, is in the $-\hat{\mathbf{z}}$ direction, as shown below. Determine the equations of motion, expressed only in terms of the cylindrical radial coordinate, $r(t)$, and the cylindrical polar coordinate, $\theta(t)$, by: a) determining the Lagrangian, L; and b) using the Hamilton's variational principle of least action and the associated Euler-Lagrange equations to derive the differential equations of motion for the r and θ particle coordinates.

Hint – note that the z coordinate should be eliminated using $r/z = \tan\alpha$, after the Lagrangian is most easily expressed in the complete cylindrical coordinates (r,θ,z).

Problem 4.

Consider the 2nd order ODE, $xy''(x)+(1-x)y'(x)+qy(x)=0$, utilize the Frobenius power series method of solution for $y(x)$. a) Determine the indicial equation and the associated lowest power, k. b) Use this indicial equation solution to determine the associated recurrence relation between successive power series coefficients. c) Determine specific values of the parameter q, which force a termination of the Frobenius series, resulting in a finite power series polynomial solution, $y(x)$, and determine the first three lowest order polynomial solutions of the ODE, called $y_0(x)$, $y_1(x)$, $y_2(x)$, by picking specific choices of q for each case.

Hint – the Frobenius power series solution is of the form $y(x)=\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$.

Physics (graduate level)

Problem 1.

Imagine two noninteracting particles, each of mass m, in the infinite square well. If one is in the state ψ_n, and the other is in ψ_m $(m \neq n)$, calculate $\langle (x_1 - x_2)^2 \rangle$, assuming (a) they are distinguishable particle, (b) they are identical bosons, and (c) they are identical fermions.

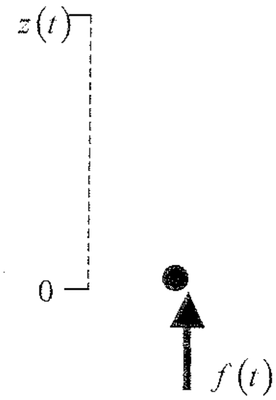
Problem 2.

We can extend the theory of a free electron gas into the relativistic domain by replacing the classical kinetic energy, $E = p^2/2m$ by the relativistic formula $E = \sqrt{p^2c^2 + m^2c^4} - mc^2$. Momentum is related to a wave vector in the usual way $\mathbf{p} = \hbar\mathbf{k}$. In particular in the extreme relativistic limit, $E \approx pc = \hbar ck$.

(a) Calculate the total energy in the ultra-relativistic regime.
(b) Write the total electron energy in terms of the radius, the number of electrons N and the mass of electrons.
(c) Calculate the gravitational energy of a uniformly dense sphere. Express your answer in terms of G (the constant of universal gravitation). Note that in this case there is no stable minimum, regardless if the total energy is positive, degeneracy forces exceed gravitational force and the star ill expand, whereas if the total is negative, gravitational forces win out, and the star will collapse. Find the critical number of nuclei N_c, such that gravitational collapse occurs for $N > N_c$. This is called a Chandrasekhar limit. The stars will not form white dwarfs, but collapse further becoming neutron stars.

Problem 3.

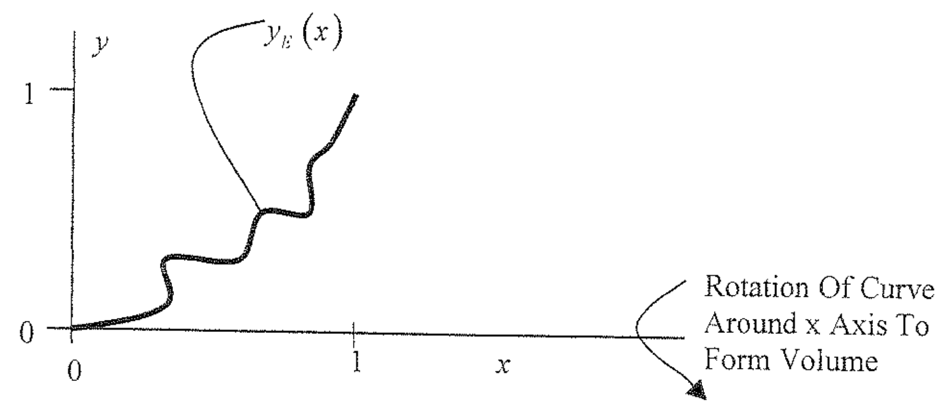
Consider an elastic medium, where a representative coordinate, z, has a natural oscillation frequency of ω, which is impulsively driven with a harmonic forcing function, $f(t)$. The objective is to determine the particular solution, $z(t)$ for all time, of the driven coordinate equation ODE, $\ddot{z}(t) + \omega^2 z(t) = f(t)$, for the initial condition $z(t) = 0, \quad t \leq 0$, where $f(t) = \begin{cases} 0, & t < 0, t > \pi/\omega \\ F\sin(\omega t), & 0 \leq t \leq \pi/\omega \end{cases}$. **a) Determine Green's function**, $G(t,t')$, for the ODE. **b) Using the Green's function, determine** the particular coordinate **solution**, $z(t)$, **for** $0 \le t \le \pi/\omega$. **c) Finally**, **determine** the **extended** coordinate **solution**, $z(t)$, **for** $t > \pi/\omega$.



Hint: 1) In b), you might consider the particular solution Green's function integral using an exponential representation of the harmonic forcing function. 2) In c), you might consider connecting the particular solution to a homogeneous solution.

Problem 4.

Consider curve, $y(x)$, from origin $(x,y)=(0,0)$ to $(1,1)$, rotated about x axis to form volume; the average cross sectional area, A, functional $J_A = \int_0^1 dx A(y,y',x)$; and the average square slope, S^2, functional $J_{S^2} = \int_0^1 dx S^2(y,y',x)$. Consider sum of functionals, as a design objective, $J = J_A + \pi J_{S^2}$, for a minimum average cross sectional area and square slope (where the π factor is added for convenience). **a) Determine precise functional, J, used to explore extremum curve solutions, $y(x)$, where $\delta J = 0$, and write down the associated Euler equation. b) Solve for extremum curve, $y_E(x)$, which satisfies boundary conditions, $(0,0)$ & $(1,1)$; draw sketch of solution; and state why this might represent a curve satisfying the extremum design objectives.**



Hint: For a general solution, consider how many independent solutions an n^{th} order ODE exhibits.

Astrophysics Qualifying Exam January 14, 2006

You must answer all the questions from the required courses plus 1 of 2 questions covering the elective courses. Look through the entire exam in order to plan how you should budget your time.

Answer all four of the following questions from the required courses.

Observational Techniques Class ASTR 5150 (calculator allowed) REQUIRED

Problem 1

a) Write a general expression for the signal to noise for a photon counting device as a function of time. Define your variables and include a short description of each source of noise that might be present.

b) In the case where you are using a photon counting device and your S/N is limited by a bright sky which is very bright compared to anything else, describe how the S/N of your point source target varies if you

i) double the telescope diameter

ii) double your spectral bandwidth

iii) double the exposure to your source

iv) double the distance to your source

v) double the radius of your photometric aperture

c) You are using a spectrograph mounted at the prime focus of an f/2 10 m telescope. The spectrograph consists of an f/2 100 mm collimator and an f/1.2 60 mm camera. The grating is a 100 lines/mm reflection grating used in first order at incident angle of a =30 degrees.

i) If the typical site seeing is 1" FWHM, how wide should you make your entrance slit to the spectrograph (in microns or mm)?

ii) If you want to Nyquist sample a typical stellar object at your detector, what would be the optimal pixel size for your ccd array?

iii) Find the angle of reflection β, for 5000 Angstrom light. Show a short derivation for an expression for the angular dispersion of your spectrum, $\delta B / \delta \lambda$, at 5000 Angstroms. What is the linear dispersion at the detector in Angstroms per pixel (use the pixel size you found above or use 25 microns.)

Interstellar Medium REQUIRED

Problem 2

A B star is immersed within a large HI cloud. The number of photons per second from the star with energy greater than 13.6 eV is $2 \cdot 10^{47}\ s^{-1}$, and the density and temperature of the HI cloud are $n_H = 10^7 m^{-3}$ and $T_{HI} = 100$ K, and gas-to-dust ratio within the cloud is 100.

a) What is the radius R_{HII} of the (initial) ionized volume surrounding the star? The total recombination coefficient is $\beta_2 = 2 \cdot 10^{-16} T_e^{-3/4} m^3 s^{-1}$. ('initial' implies do not concern yourself with the initial pressure imbalance between HII and HI regions)

b) Consider two dust grains with Grain A at a distance $2R_{HII}$ from the star and Grain B at a distance $4R_{HII}$ from the star. How do the grain temperatures T_A and T_B qualitatively compare to T_{HI}? Are they comparable to T_{HI}, much hotter or much colder?

c) Explain your reasoning for part b).

d) What is the ratio T_A / T_B?

e) Compute how fast Grain A spins, and how fast Grain B spins. Assume that the grains are spherical with a radius of 100 mm and a density of 2000 kg m^{-3}.

Cosmology REQUIRED

Problem 3

In the Big Bang theory, the early epochs prior to recombination are thought to be radiation dominated. The pressure of a photon gas in terms of it energy density is $p = 1/3\varepsilon$, and the inertial mass density of the radiation ρ_r is related to its energy density by $\varepsilon = \rho_r c^2$. For this radiation-dominated epoch, please answer the following questions.

a) How does the number density of photons vary with the scale factor R and the redshift z?

b) How does the energy density of photons depend on R and z?

c) For black –body radiation, the energy density can be written in terms of the temperature. How does the temperature depend on R and z?

d) For universes with $\Lambda = 0$, and ignoring the curvature terms that can be neglected at early epochs, the Einstein field equations are:

$$\ddot{R} = -\frac{4\pi GR}{3}(\rho + \frac{3p}{c^2})$$

$$\dot{R}^2 = \frac{8\pi G\rho}{3}R^2$$

Please determine from these equations how the scale factor R and the energy density ε depend on time.

e) Finally, combine these results to say how the scale factor R evolves with time t in the radiation-domination epoch.

Stellar Atmospheres REQUIRED

Problem 4

a) The solar constant is 1.38×10^6 ergs per cm^2 per sec. The angular diameter of the Sun is 32 arc minutes. Calculate the effective temperature of the Sun.
b) Assume that this is the Wein Temperature. What is the wave length of maximum radiation?
c) The simplified energy level diagram for sodium is given below where the energy level and the electronic configuration follow the normal designations. Using the effective temperature calculated in part "a" determine the relative populations of the 3.6-ev state and the 3.2-ev state. (combine the two D 5/2 and D 3/2 states of the 3.6-ev state for this calculation)
d) The Sun is a G2 V star with strong ionized calcium lines. Show this to be the case. The ionization potential for calcium is 6.1 ev. Typical gas and electron pressures for the Sun are 4.75 and 1.35 in the log at optical depth = 2/3.

Sodium energy configuration:

$D_{5/2}$

>--------- 3.6 ev

$D_{3/2}$

$S_{1/2}$ ---------- 3.2 ev

$P_{3/2}$

>--------- 2.1 ev

$P_{1/2}$

$S_{1/2}$ --------- 0.0 ev

OPTIONAL Galaxies/Milky Way

Problem 5

Please refer to the accompanying figure showing data based on a longslit spectrum of an edge-on spiral galaxy with the slit aligned with the major axis of the galaxy. Plotted are the observed wavelength in Angstroms of the H α emission line against the position in arcseconds. Assume that the motion of the Sun and Milky Way have been corrected for already. Based on the data in this figure, estimate the mass of this galaxy. Please state all assumptions /steps you make in order to do this. Compare your answer quantitatively to an estimated mass of the Milky Way interior to the Solar circle base on the orbit of the sun.

OPTIONAL Stellar Interiors

Problem 6

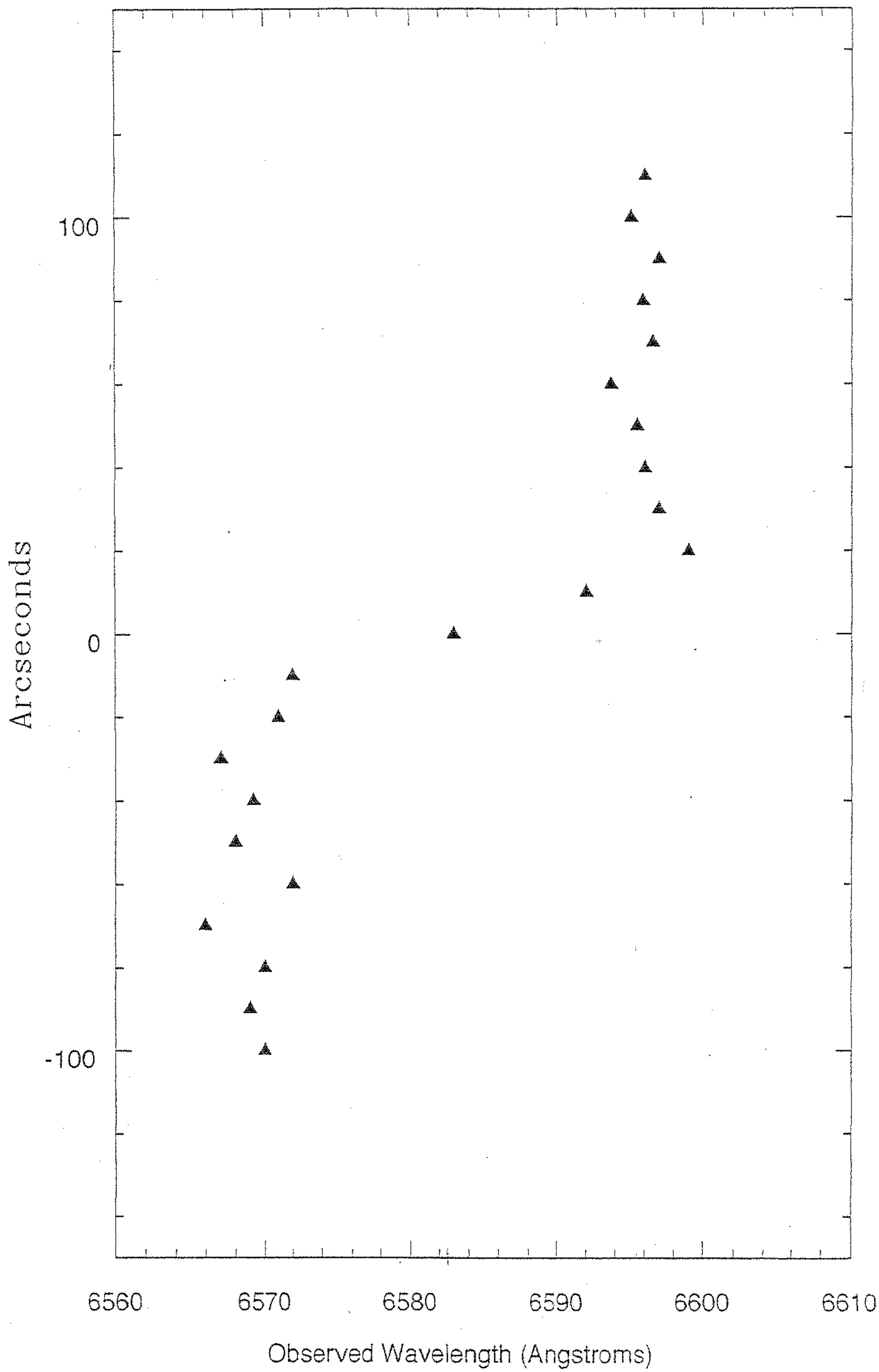
From basic stellar interior principle estimate the follow time scales using the sun as a model and your general knowledge of stellar properties. Show your work and state all of your assumptions.

a) A dynamical, free fall time scale for the Sun where the total internal support of the Sun was instantaneously removed.

b) Nuclear fusion time scales for all stars where the total energy of a star is supported by nuclear burning of H to He in the star core.

c) The Kelvin-Hemholtz gravitational collapse time scale for all stars.

d) The pulsation time scale of the sun.



100
0
-100
Arcseconds
6560
6570
6580
6590
6600
6610
Observed Wavelength (Angstroms)

Physics (graduate level)

Problem 1.

Imagine two noninteracting particles, each of mass m, in the infinite square well. If one is in the state ψ_n, and the other is in ψ_m $(m \neq n)$, calculate $\langle (x_1 - x_2)^2 \rangle$, assuming (a) they are distinguishable particle, (b) they are identical bosons, and (c) they are identical fermions.

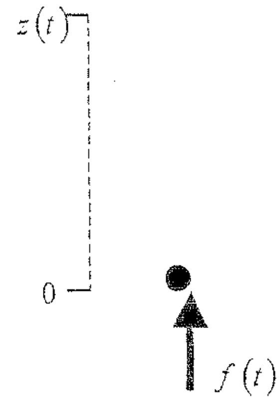
Problem 2.

We can extend the theory of a free electron gas into the relativistic domain by replacing the classical kinetic energy, $E = p^2/2m$ by the relativistic formula $E = \sqrt{p^2c^2 + m^2c^4} - mc^2$. Momentum is related to a wave vector in the usual way $\mathbf{p} = \hbar\mathbf{k}$. In particular in the extreme relativistic limit, $E \approx pc = \hbar ck$.

(a) Calculate the total energy in the ultra-relativistic regime.
(b) Write the total electron energy in terms of the radius, the number of electrons N and the mass of electrons.
(c) Calculate the gravitational energy of a uniformly dense sphere. Express your answer in terms of G (the constant of universal gravitation). Note that in this case there is no stable minimum, regardless if the total energy is positive, degeneracy forces exceed gravitational force and the star ill expand, whereas if the total is negative, gravitational forces win out, and the star will collapse. Find the critical number of nuclei N_c, such that gravitational collapse occurs for $N > N_c$. This is called a Chandrasekhar limit. The stars will not form white dwarfs, but collapse further becoming neutron stars.

Problem 3.

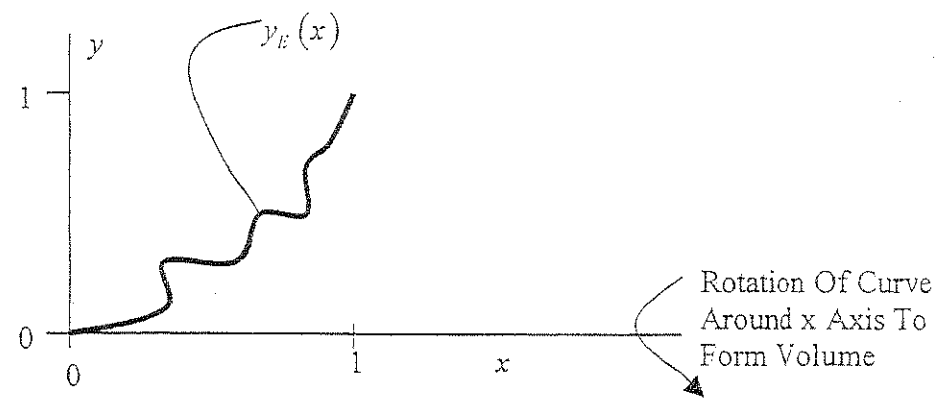
Consider an elastic medium, where a representative coordinate, z, has a natural oscillation frequency of ω, which is impulsively driven with a harmonic forcing function, $f(t)$. The objective is to determine the particular solution, $z(t)$ for all time, of the driven coordinate equation ODE, $\ddot{z}(t) + \omega^2 z(t) = f(t)$, for the initial condition $z(t) = 0, \quad t \leq 0$, where $f(t) = \begin{cases} 0, & t < 0, t > \pi/\omega \\ F\sin(\omega t), & 0 \leq t \leq \pi/\omega \end{cases}$. **a) Determine Green's function,** $G(t,t')$, for the ODE. **b) Using the Green's function, determine** the particular coordinate **solution**, $z(t)$, **for** $0 \le t \le \pi/\omega$. c) **Finally**, **determine** the **extended** coordinate **solution**, $z(t)$, **for** $t > \pi/\omega$.



Hint: 1) In b), you might consider the particular solution Green's function integral using an exponential representation of the harmonic forcing function. 2) In c), you might consider connecting the particular solution to a homogeneous solution.

Problem 4.

Consider curve, $y(x)$, from origin $(x, y) = (0,0)$ to $(1,1)$, rotated about x axis to form volume; the average cross sectional area, A, functional $J_A = \int_0^1 dx A(y, y', x)$; and the average square slope, S^2, functional $J_{S^2} = \int_0^1 dx S^2(y, y', x)$. Consider sum of functionals, as a design objective, $J = J_A + \pi J_{S^2}$, for a minimum average cross sectional area and square slope (where the π factor is added for convenience). **a) Determine precise functional, J, used to explore extremum curve solutions, $y(x)$, where $\delta J = 0$, and write down the associated Euler equation. b) Solve for extremum curve, $y_E(x)$, which satisfies boundary conditions, $(0,0)$ & $(1,1)$; draw sketch of solution; and state why this might represent a curve satisfying the extremum design objectives.**



Hint: For a general solution, consider how many independent solutions an n^{th} order ODE exhibits.

Astrophysics Qualifying Exam January 15, 2005 (afternoon)

You must answer all of the questions from the required courses plus 2 additional questions covering the elective courses. Look through the entire exam in order to plan how you should budget your time.

Answer all four of the following questions from the required courses.

Astronomical Techniques (required):

Problem 1. You want to observe a galaxy with a flux of 10^{-16} erg/s/cm^2/A at 5000 A using a V-filter.

a) What is the corresponding flux in photons/s/cm^2/A?

b) What is the corresponding flux in erg/s/cm^2/Hz?

c) How wide is a typical V-filter?

d) Below are listed a set of telescope and instrument combinations. Assume you are using a photon counting detector and that all of the galaxy's flux is contained within 4 pixels and that the image scale is 1"/pixel. Which of the two telescope/instrument combinations would achieve a signal-to-noise (S/N) of 100 in the least amount of time? How long would it take?

e) If the galaxy had the same luminosity but were twice as far away, how long would it take to reach a S/N of 100?

Configuration	A	B
Aperture	4m	2.5m
Teles. Efficiency	0.9	0.95
Dark Current	0	0.001 (e-/pix/sec)
Read Noise	5.0	2.0 (e-/pix)
V Sky brightness	20	21 (mag/arsec^2)
Filter Efficiency	0.80	0.80

Note: the zero-point for the V-band is $F_\lambda = 3.37 \times 10^{-9}$ erg/s/cm/A

Stellar Atmospheres (required):

Problem 2. The equation of Hydrostatic Equilibrium is given by the expression

$$dP / dz = -g\rho$$

a) Rewrite this equation in a form used in stellar atmospheres relating the pressure, optical depth, opacity and gravity.

b) Assume that the opacity κ is directly proportional to the gas pressure, P_g. Show how the gas pressure, P_g, and the opacity, κ, would vary with optical depth, τ.

c) For a given optical depth show how the gas pressure varies with gravity.

d) If the temperature distribution follows the relation

$$T^4 = \tfrac{3}{4}\, T_{eff}^{\ 4}\, (\tau + 2/3)$$

Then show that temperature will be constant for small optical depths but in deeper layers the temperature will be directly proportional to the square root of the gas pressure, i.e.

$$T \alpha \sqrt{P_g}$$

Interstellar Medium (required):

Problem 3. Suppose you observe a background continuum radio source (such as a quasar) of brightness temperature T_0 which is behind three different absorbing clouds with source temperature T_1, T_2, T_3 and optical depth τ_1, τ_2, τ_3 (see figure).

a) Write down an expression for the brightness temperature, T_b, observed at the Earth as a function of the seven parameters T_0, T_1, T_2, T_3 and τ_1, τ_2, τ_3. If you cannot quantify, qualitatively explain the solution.

b) Evaluate your solution for $\tau_3 >> 1$. Comment on the result.

c) Evaluate your solution for τ_1 and $\tau_2 << 1$. Comment on the result.

d) A large cloud of dust is now placed between Earth and Cloud #3. The dust cloud causes 1.0 magnitudes of extinction at V-band wavelengths. Quantitatively and qualitatively explain the resulting impact on the observed spectrum.

Galaxies and Cosmology #1 (required):

Problem 4. What is the approximate expansion timescale at the epoch of recombination? That is, what is 1/H? Recall that the expansion equation for a Friedman-Lemaitre universe with zero cosmological constant is given by:

$$H^2 = (\dot{R}/R)^2 = 8\pi\rho G/3 - kc^2/R^2,$$

Where k is the curvature (-1, 0, +1), and R is the scale factor. Be sure to state your assumptions.

Answer two additional questions from the following elective courses.

Stellar Interiors #1 (optional):

Problem 5. The opacity of the Sun at its center implies a mean free path for a photon of approximately 0.5 cm. From random walk theory we know that the rms square distance $< X_N^2 >$ of N interaction with step size l is $< X_N^2 > = N\, l^2$. Using this information and from first principles of stellar interiors

a. Calculate the central temperatures and pressure at the center of the Sun.

b. Calculate the mean time it takes a photon to travel from the center to the surface.

c. From this information i.e. your calculated central temperature, pressure and the mean travel time, calculate the luminosity of the Sun. Compare this value with the known observed solar luminosity.

$\frac{GMm}{r^2} = N = \frac{kg\,m}{s^2}$

$[G] = \left(\frac{1}{kg}\right)\left(m^3\right)\left(\frac{1}{s^2}\right)$

$[P] = \frac{[F]}{[A]} = \frac{\frac{kg\,m}{s^2}}{m^2} = \frac{kg}{s^2\,m}$

Stellar Interiors #2 (optional):

Problem 6. The opacity and energy generation inside a star can be given by the following relationships:

$$\kappa = \kappa_0 \rho / T^{3.5} \quad \text{and} \quad \varepsilon = \varepsilon_0 \rho T^{17} .$$

For homologous stars with the same chemical composition but differing mass show that the mass-luminosity relation and the mass-effective temperature relations are of the form

$$L_* \propto M_*^{67/13} \quad \text{and} \quad L_* \propto T_{eff}^{268/49} .$$

Stars and the Milky Way (optional):

Problem 7. A simple model for the density distribution of a spherical galaxy, and which is generally a good representation, is given by:

$$\rho(r) = C/(r_0^2 + r^2)$$

where C and r_0 are constants.

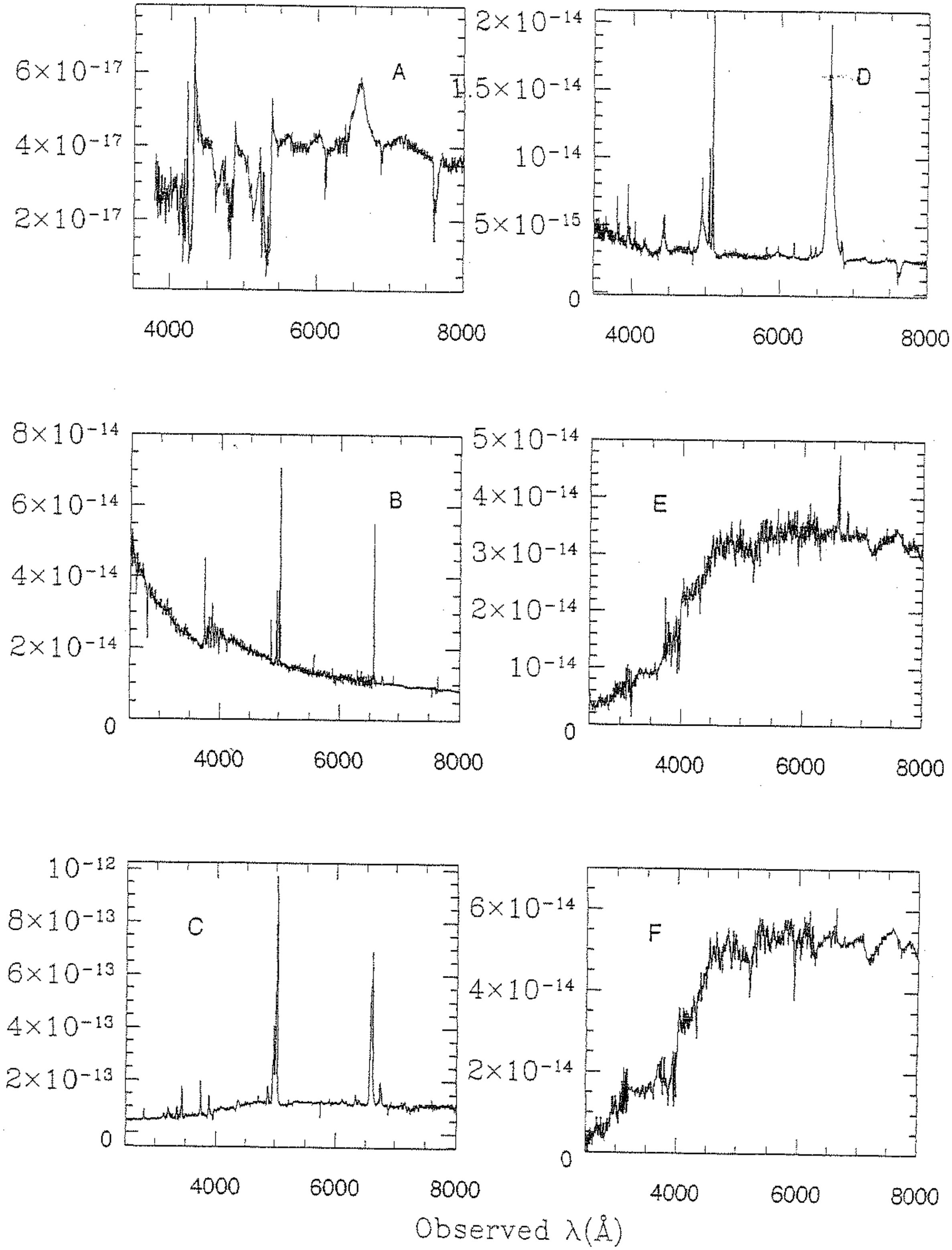
a) Qualitatively discuss the rotation curve that would correspond to this model in the limits of $r \ll r_0$ and $r \gg r_0$ (**Hint:** recall the equation for mass conservation from stellar interiors).

b) Assuming that this is a good model for the density distribution within the Milky Way galaxy, derive a corresponding expression for the integrated mass distribution (i.e. the mass within a given radius r).

c) If $r_0 = 2.8$ kpc and $C = 4.6 \times 10^8$ $M_\odot$/kpc compute the total amount of mass within the radius of the Sun's orbit about the Galactic center. Repeat you calculation for a radius of 25 kpc and compare these results.

Galaxies and Cosmology #2 (optional):

Problem 8. The figure below shows six spectra of extragalactic sources. The x-axis is the observed wavelength in Angstroms. The y-axis is the observed flux in units of ergs/s/cm/A. Classify each one as best you can and explain your reasoning. Try to be as specific and quantitative as possible – guesses without correct reasoning won't earn credit. In particular, you will want to identify specific spectral features and explain their significance.


A
6×10⁻¹⁷
4×10⁻¹⁷
2×10⁻¹⁷
D
2×10⁻¹⁴
1.5×10⁻¹⁴
10⁻¹⁴
5×10⁻¹⁵
0
B
8×10⁻¹⁴
6×10⁻¹⁴
4×10⁻¹⁴
2×10⁻¹⁴
0
E
5×10⁻¹⁴
4×10⁻¹⁴
3×10⁻¹⁴
2×10⁻¹⁴
10⁻¹⁴
0
C
10⁻¹²
8×10⁻¹³
6×10⁻¹³
4×10⁻¹³
2×10⁻¹³
0
F
6×10⁻¹⁴
4×10⁻¹⁴
2×10⁻¹⁴
0
4000
6000
8000
Observed λ(Å)

2004

Day 2

Morning

Statistical Mechanics

Problem 1

An ideal gas of N spin-$1/2$ fermions is confined to a volume V. Calculate the zero-temperature limit of (a) the chemical potential, (b) the average energy per particle, (c) the pressure, (d) the Pauli spin susceptibility. Assume that each fermion interacts with an external field in the form: $2\mu_B H S_z$, where μ_B is the Bohr magneton.

Quantum Mechanics

Problem 1

A particle is placed into an infinite potential well of the width a ($0<x<a$) in the ground state. At some moment the wall of the well is moved to the point b ($b>a$) for a short period of time. Find the probability of excitations to different stationary states after the motion of the wall was stopped. Find the validity condition of obtained results. In particular case consider $b = 2a$.

Relativity

Problem 1

Proton with $\gamma = 1/\sqrt{1-(v/c)^2}$ collides elastically with a proton at rest. If two protons rebound with equal energies, what is the angle θ between them?

Electricity and Magnetism

Problem 1

Write down Maxwell's equations in a non-conducting medium with constant permeability and susceptibility ($\rho = \mathbf{j} = 0$). Show that **E** and **B** each satisfy the wave equation, and find an expression for the wave velocity. Write down the plane wave solutions for **E** and **B** and show how **E** and **B** are related.

Problem 2

Discuss the reflection and refraction of electromagnetic waves at a plane interface between the dielectrics and derive the relationships the angle of incidence, refraction, and reflection (see problem 1).

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

Astronomy Qualifying Exam Spring 2004

Interstellar Medium and Diffuse Matter (required)

1) Consider a hot star just formed within a typical cloud of neutral Hydrogen (HI region) of density $n_0 = 10^7 \text{ m}^{-3}$ which subsequently ionizes a spherical volume within the cloud (i.e., the Stromgren Sphere).

a) Roughly, how does the number density inside the HII region compare to the number density in the surrounding HI region? What is the physical reason for this particular ratio?

b) Roughly, how does the temperature inside the HII region compare to that in the surrounding HI region?

c) What is the ratio of gas pressures for the HII region relative to the HI region and what does the pressure imbalance imply will happen to the nascent (just-born) HII region?

d) Derive an expression for the initial radius of the Stromgren sphere in terms of the important properties of the HI region and the hot star.

e) Assume that after a very long time the Stromgren radius stabilizes. Compute the ratio of the final Stromgren radius to that of the initial Stromgren radius.

f) Compute the ratio of the mass of the final Stromgren sphere to that of the initial Stromgren sphere.

g) Now suppose that the star goes supernovae. What will happen to the radius of the ionized volume and why?

h) Suppose that a shock moves outward from the supernovae at 500 km/sec and that the shocked gas is at a temperature of 10^4 °K. What is the Mach number (M) of the shock?

i) Suppose that when the shock front has a radius R = 1 pc, it has swept up all the interstellar material within R, and that the ratio of the density within the swept-up shell ρ_2 to that in the surrounding ISM ρ_0 can be approximated by the square of the Mach number M. Derive an expression for the thickness ΔR of the swept-up region. Evaluate your expression.

j) Assuming no other external influences, the ionized gas will eventually cool down. What is the most likely mechanism by which it will cool (be specific)?

Astronomical Techniques (required)

1) Suppose you wish to do stellar photometry with a CCD detector placed at the focus of an astronomical telescope.

a) Write a paragraph describing the fundamental detection mechanism of a CCD. Include the basic physics as well as a description of the physical structure of a pixel and the mechanisms used to read out the pixel array. Include a diagram showing the process.

b) Write a general expression describing the signal to noise within a single pixel of a CCD as a function of integration time (t) assuming that all the light from a star falls upon a single pixel. **Hint:** Be sure to identify and include all the possible noise sources that will affect the signal to noise of the observation.

c) For a typical CCD device, estimate the maximum signal to noise achievable with a single exposure on a stellar source. Assume that the PSF is critically sampled and assume that you have a perfect detector with noiseless electronics.

d) Estimate the minimum number of photons required for a extraterrestrial observer to detect the transit of Earth across the Sun with a 5-sigma level of significance. Assume a perfect, noiseless detector. Be sure to identify any assumptions that you make.

e) Use simple arguments to estimate the duration of the transit of Earth across the disk of the Sun as seen from a nearby star/planetary system.

Choose <u>Four</u> Additional Problems from the Following Elective Courses

Note that any combination of 4 problems is acceptable. However, you must indicate which 4 problems you want to be graded.

Galaxies and Cosmology (optional)

1) The "Cosmological Time-Scale Problem" results from the discrepancy between measurements of the Hubble Constant and the age of the Universe. Recall that the expansion equation for a Friedmann-Lemaitre universe is given by:

$$H^2 = (\dot{a}/a)^2 = 8/3\ \pi\rho G + 1/a^2R^2$$

a) Show that this equation is essentially a statement of energy conservation.

b) Show that for a given measured value of the Hubble Constant (H_o), the age of an open, low-density universe is up to 50% longer than the age of a matter-dominated, critical universe.

c) Discuss the current status of the measurements of the Hubble Constant and why these measurements appear to be in conflict with the measured ages for Globular Clusters. What is the current thinking on how this discrepancy might be resolved? What additional observation evidence exists that supports this interpretation?

2. The Oort constants A and B are used to characterize the Galactic rotation in the solar neighborhood. They are readily obtained from measuring the velocities of stars as a function of Galactic longitude *l*. The relevant definitions are:

$A = -1/2[d\theta/dR|_{R0} - \theta_0/R_0]$ $B = -1/2[d\theta/dR|_{R0} + \theta_0/R]$

and the radial and transverse velocities of nearby stars as a function of Galactic longitude can be written then as $v_r = Ad\sin(2l)$, and $v_t = Ad\cos(2l) + Bd$, where d is the distance. $R_0 = 8$ kpc and $\theta_0 = 220$ km/s are the local parameters of Galactic rotation. Measurements of the radial and transverse velocities indicate that A = 14.4 +- 1.2 km/s/kpc, B = -12.0 +- 2.8 km/s/kpc. Evaluate the Oort constants for the cases of Keplerian rotation and flat rotation. Discuss your answer.

Radiative Processes and Stellar Atmospheres (optional)

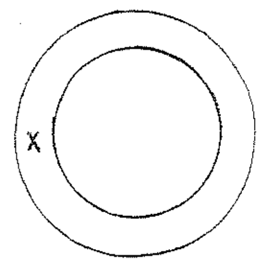
1) There are at least 7 different temperature measurements that are used in describing a stellar atmosphere. For each one of these temperatures

a) Give the definition for each temperature and explain the physical principle(s) which defines each temperature.

b) List the observational data necessary to make this measurement.

c) Write an essay (limit it to less than 500 words) describing the relationship and importance the ***most significant*** of these temperature measurements ***have*** with respect to our general understanding and models of a stellar atmosphere. You should include in your essay how each temperature relates to our knowledge of the properties of a standard stellar atmosphere and its theoretical model.

2. In stellar atmospheres the transfer equation for emerging radiation is



$\mu\, dI/d\tau = I - S$ where $\mu = \cos\theta$.

a) If the source function is given as $S = a_0 + a_1 \tau$, show that the emergent intensity is also a linear function of $\cos\theta$.

b) Now consider a simple stellar model with a stellar surface radiating as a black body of temperature T_{surf}. Surrounding the star is a spherical shell/layer of temperature T_{layer} and thickness X. The material in this layer consists of a two level atom from which bound-bound transitions take place at a line center λ_0. In this case $T_{surf} > T_{layer}$.

c) Write a general expression for the intensity of the radiation from the stellar surface?

d) Write a general expression for the intensity of the radiation from the thin layer?

e) Write a general expression for the intensity of the radiation for the combined stellar surface and thin layer significantly away from the line center λ_0, and at a line center λ_0.

f) Consider the optically thick case $\tau >> 1$: what is the expression of the emergent intensity coming from the combined stellar surface and thin layer significantly away from the line center λ_0, and at a line center λ_0.

g) Consider the optically thin case $\tau << 1$: what is the expression of the emergent intensity coming from the combined stellar surface and thin layer significantly away from the line center λ_0, and at a line center λ_0.

h) Describe the progression of the line profile (by words or by a drawing) as the optical depth changes from optically thin, through $\tau = 1$, to the optically thick case.

i) As you are looking at the center of the star, draw a center-to-limb intensity profile for this star and spherical ring. Do this for a wavelength significantly away from the line center λ_0, and then at a line center λ_0. Explain your drawing.

Stellar Interiors (optional)

1. The opacity of the Sun at its center implies a mean free path for a photon of approximately 0.5 cm. From random walk theory we know that the rms square distance $< X_N^2 >$ of N interaction with step size l is $< X_N^2 > = N l^2$. Using this information and from first principles of stellar interiors:

a) Calculate the central temperature and pressure at the center of the Sun.

b) Calculate the mean time it takes a photon to travel from the center to the surface.

c) From this information i.e., your calculated central temperature, pressure and the mean travel time, calculate the luminosity of the Sun. Compare this value with the known observed solar luminosity.

2. The opacity of a star's interior composition has the form $\kappa = \kappa_o \rho^\alpha T^\beta$ where α and β are slowly varying functions of density and temperature.

For stars with low temperatures and low to intermediate masses, the opacity is primarily from bound-bound transitions where $\alpha = ½$ and $\beta = 4$.

For stars with intermediate temperatures and intermediate to high mass, the opacity is primarily from bound-free and free-free transitions where $\alpha = 1$ and $\beta = -3.5$.

For stars with extremely high temperatures and very high masses, the opacity is primarily from free-free transitions and electron scattering where $\alpha = \beta = 0$.

From first principles, i.e., the equations of stellar structure, show that

$L \sim M^{5.5} R^{-0.5}$ for low to intermediate mass stars
$L \sim M^3$ for intermediate to high mass stars, and
$L \sim M$ for very high mass stars.

Stars and the Milky Way (optional)

1) Consider a galaxy cluster with a spherically symmetric mass density profile of the form:

$$\rho(r) = \rho_0 \,(r/r_0)^{-2}$$

a) Suppose that a spherical galaxy of radius R_g is in circular orbit of radius r about the cluster center; assume that $R_g << r$. Calculate the magnitude and direction of the *tidal* acceleration a_t at the point in the galaxy nearest the cluster center and at the point in the galaxy farthest from the cluster center. Draw a figure showing the direction of the gravitational and tidal accelerations due to the cluster at the galaxy center, at the near and far points, and at points also at a distance R_g from the center of the galaxy that are 90^0 from the near and far points. Explain the reason for the direction of the tidal accelerations at each location. Also explain the dependence of a_t on the various cluster and galaxy parameters. Note that the cluster should be treated as an *extended* mass distribution M(r), rather than as a point mass

b) Suppose that the radius R_g of the galaxy is determined by the condition that the tidal acceleration at this radius is equal to the gravitational binding acceleration due to the mass of the galaxy, M. Calculate R_g in this case and show that $\langle\rho(R_g)\rangle = \langle\rho_{cl}(r)\rangle$, where $\langle\rho(r)\rangle$ is the mean density within r. Is $\langle\rho_{cl}(r)\rangle$ a lower or upper limit for the mean density of a galaxy in circular orbit at r? Explain.

2) Consider the Virial Theorm, specifically in the case where N identical galaxies, each of mass M, merge into a single galaxy.

a) Determine how the radius, velocity dispersion, and surface brightness of the merged galaxy compares with those of its precursors. Define the radius, R_i, of the N individual precursor galaxies so that each has a gravitational binding energy $U_i = GM^2/R_i$, and a total energy $E = -GM^2/2R_i$. Define the mean surface brightness of the N pre-merger galaxies as $\Sigma_i = M/R_i^2$. Assume that the total energy of the merger is equal to the sum of the N constituents (i.e., neglect any orbital energy). Assuming that the new galaxy differs from its constituent parts only by scaling factors then we can define a new radius R_N in the same way as its precursors.

What is the new radius, R_N, in terms of R_i?
What is Σ_i in terms of Σ_N?
What is the new r.m.s velocity dispersion, σ_N, in terms of σ_i?

b) Now use the Virial Theorm to explain the origin of the Faber-Jackson relation for Elliptical Galaxies, which relate the luminosity of a galaxy to its stellar velocity dispersion.

Part II: Section B

Do <u>any two</u> of the three problems in Section B.

5. **Statistical Mechanics** Calculate the electric polarization P of an ideal gas at temperature T, consisting of molecules having a constant electric dipole moment p, when the gas is placed in a homogeneous external electric field E.

6. **Advanced Quantum Mechanics.**

Drawing on results from graduate quantum mechanics, we have commutation relations for the Hermitian generators $\tilde{J}_x$, $\tilde{J}_y$, and $\tilde{J}_z$ representing the components of the angular momentum operator:

$$[\tilde{J}_x, \tilde{J}_y] = i\hbar \tilde{J}_z$$

$$[\tilde{J}_y, \tilde{J}_z] = i\hbar \tilde{J}_x$$

$$[\tilde{J}_z, \tilde{J}_x] = i\hbar \tilde{J}_y$$

Using these relationships show that the simultaneous eigenvector for the operator $\tilde{J}^2$ and $\tilde{J}_z$ have eigen values $j(j+1)\hbar$ and $m\hbar$ respectively.

7. **Electricity and Magnetism** Determine the relativistic motion (the trajectory) of a charge in a system where there are parallel and uniform electric and magnetic fields directed along the z axis. Find x, y, and z as a function of time.

3. **Electricity & Magnetism** In free space, for a given volume current density $\vec{J}(\vec{r}, t)$, using the Lorentz gauge, the magnetic vector potential $\vec{A}(\vec{r}, t)$ is given by

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \int dt' \, \frac{\vec{J}(\vec{r}\,', t')}{|\,\vec{r} - \vec{r}\,'\,|} \, \delta(t' + \frac{|\,\vec{r} - \vec{r}\,'\,|}{c} - t) \quad .$$

Assume the sinusoidal time dependence for all sources is given by

$$\vec{J}(\vec{r}, t) = \vec{J}(\vec{r}) e^{-i\omega t} \quad ,$$

$$\rho(\vec{r}, t) = \rho(\vec{r}) e^{-i\omega t} \quad .$$

A) Show $\vec{A}(\vec{r}, t)$ may be written

$$\vec{A}(\vec{r}, t) = \vec{A}(\vec{r}) \, e^{-i\omega t}$$

where

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') e^{ik|\vec{r} - \vec{r}\,'|}}{|\,\vec{r} - \vec{r}\,'\,|} d^3r'$$

and

$$k = \omega / c \quad .$$

B) If $\vec{J}(\vec{r})$ is confined to a finite region of maximum diameter d, show that for $d \ll \lambda \ll r$, where $\lambda = 2\pi c/\omega = 2\pi k$,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int \vec{J}(\vec{r}\,') \, e^{i\vec{k} \cdot \vec{r}\,'} d^3r'$$

where $\vec{k} = k\hat{r}$. Give the physical interpretation of this form for $\vec{A}(\vec{r}, t)$.

C) Use $kd \ll 1$, the identity

$$\int \vec{J}(\vec{r}\,') \, d^3r' = - \int \vec{r}\,'(\vec{\nabla}\,' \cdot \vec{J}) \, d^3r' \quad ,$$

and the continuity equation in the result of b) to show that

$$\vec{A}(\vec{r}, t) = \frac{-i\mu_0 \omega}{4\pi} \, \vec{P} \, \frac{e^{i(kr - \omega t)}}{r}$$

where $\vec{P}$ is the electric dipole moment of the charge distribution as defined in electrostatics.

D) Show how to determine $\vec{E}$ and $\vec{B}$ for the $\vec{A}(\vec{r}, t)$ given in c).

4. **Quantum Mechanics** In the Born approximation determine the amplitude and cross section of scattering of particles in the potentials $U(r)$ shown below. Find the limiting cases of small and large energies. Obtain the validity conditions for each case.

A) $U(r) = \alpha / r^2$

B) $U(r) = U_0 \, e^{-r/R}$

~~C) $U(r) = (\alpha/r) \, e^{-r/R}$~~

Part II: Section A

Do <u>each</u> of the four problems in Section A.

1. **Mathematical Methods.**

   A) Estimate the following integral in the limits $a \gg b$ and $a \ll b$

$$\int_0^\infty \frac{\sin(x/a)}{x(x^2+b^2)}dx \quad .$$

   Note that

$$\int_0^\infty \frac{\sin(y)}{y} = \frac{\pi}{2} \quad \text{and}$$

$$\int_0^\infty \frac{dy}{y^2+1} = \frac{\pi}{2} \quad .$$

   B) Determine <u>divergence</u> and <u>curl</u> of the following vectors: $(\vec{a}\cdot\vec{r})\vec{b}$ and $(\vec{a}\times\vec{r})$.

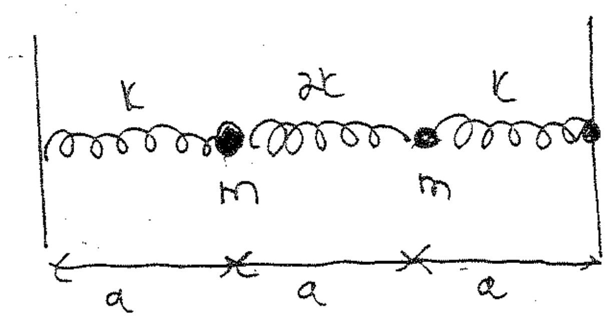
   Note: Make use of the following properties:

$$\delta_{i\,i} = 3 \quad \text{and}$$

$$\frac{\partial x_l}{\partial x_i} = \delta_{i\,l}$$

   where two identical running indexes imply a summation over that index.

2. **Classical Mechanics.** Two particles each of mass m move in one dimension at the junction of the massless springs shown. The middle spring has force constant $2K$ and the end springs have force constant K. All springs have relaxed length a. Write the Lagrangian and find the eigenfrequencies and normal modes of the system.



University of Wyoming
Department of Physics and Astronomy
Written Preliminary Examination – Part II.
Saturday, January 16, 1999
9:00 am – 2:00 PM
Instructions for Part II
PHYSICS

1. Do six (6) problems as instructed on the following pages.
2. Answer each problem on a separate set of paper.
3. Enter the identification number assigned to you on the upper right hand corner of each answer page. Do not write your name on these pages.
4. Some integrals, numerical constants, and mathematical expressions which might be useful are provided on the data pages at the end of the examination.
5. Each problem will contribute equally to your grade on this part of the examination.
6. Partial credit will be given for orderly progress towards a correct solution of a problem; however no credit will be given for an illogical, disorganized presentation or for irrelevant material

Part I: Section B

Do any two (2) of the three (3) problems in Section B, problems 6 through 8.

6. **Atomic & Molecular Physics** Consider the Hamiltonian $H = H_0 + H'$ where H' is a small perturbation to H_0. Let the basis states of H_0 be denoted by $\langle n \mid$, so that

$$H_0 \mid n\rangle \;=\; E_n \;\mid n\rangle \;.$$

In terms of this unperturbed basis, the time dependent wavefunction for this system can be written as the expansion

$$\mid \Psi\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} \;\mid n\rangle$$

where the $c_n(t)$ are slowly varying in time.

Write down the time dependent Schrödinger equation for this wavefunction and carry out the algebra to find an expression for $c_j(t)$ in terms of $c_j(t=0)$ and an integral over t.

7. **Solid State** Consider a Fermi gas of electrons existing in a two-dimensional system, instead of the three-dimensional system you are probably more familiar with. Show that the chemical potential of this Fermi gas is given by:

$$\mu(T) = k_B T ln\left[exp(\pi n\hbar^2/mk_BT) - 1\right] \quad ,$$

where n is the number of electrons per unit area. Note: The density of orbitals of a free electron gas in two dimensions is independent of energy:

$$D(\epsilon) = m/\pi\hbar^2 \quad ,$$

8. **Nuclear Physics** *Draft note: Following is old question, as placeholder. Decide whether to ask one this time. If not change instructions.* An unstable particle of rest energy 1000 MeV decays into a muon ($m_o c^2 = 100$ MeV) and a neutrino. The mean life of the unstable particle, when at rest, is 10^{-8} seconds.

   A) Assume the particle is not at rest, but has a momentum of 1000 MeV/c. Calculate the *mean decay distance* the unstable particle will travel.

   B) In the laboratory frame what is the energy of the muon, if it is emitted at an angle of 15° to the path of the original unstable particle.

5. **Optics** One of the most fundamental aspects of physical optics is knowing how to coherently add together two waves of the same frequency but of different amplitudes and phase. Consider a point source emitting a wave described by amplitude $E(t) = A_0 e^{i\omega t}$ where A_0 is real. An experimental apparatus directs at constant velocity c fractions f_1 and f_2 of the emitted wave amplitude along two separate paths of length d_1 and d_2 before coherently recombining the waves and sending the result to a detector.

   Calculate a) the wave amplitude and b) the intensity amplitude, of the resulting wave at the detector as a function of the path difference $d = d_1 - d_2$ and other necessary parameters. Remember that the intensity must be a real function. Next c) sketch the net intensity amplitude at the detector as a function of d for $f_1 = f_2$.

Part I: Section A

Do each of the five (5) problems in Section A, problems 1 through 5 inclusive.

1. **Classical Mechanics.** A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α (> 0) are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved? Justify your answers to the latter two questions in terms of the equations you have derived.

2. **Electricity & Magnetism** Given that the potential due to a collection of point charges has the form

$$V = \sum_{i=1}^{N} V_i = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

   A) Calculate the potential at an arbitrary point along the z axis of a circular plastic disk located in the x-y plane and having radius R. One surface of the disk has a uniform charge density (surface charge density) σ.

   B) Show in the special case $z \ggg R$ that the solution reduces to that expected from a point charge.

3. **Quantum Mechanics** Consider a particle of mass m interacting in one dimension with a potential of the form

$$V(x) = -C\,\delta(x)$$

   where C is a positive constant. In effect there is a deep narrow "hole" at position zero. Assume that this system does have a bound state and find the wavefunction $\psi(x)$ for that state. (If you think there is more than one state possible, find the wave function for the ground state.) Note that the key is to properly understand how to normalize the wavefunction and also how to properly match solutions of $\psi(x)$ for $x < 0$ and $x > 0$. Also note that the energy of the bound state will be $E < 0$, which is allowed since part of space has potential $V(x) < 0$. Specify your final answer for $\psi(x)$ entirely in terms of m, C, $\hbar$, and whatever intermediate constants you define based on those. Also specify the energy of the state using these same constants.

4. **Thermodynamics** *Draft note: This is identical to one given last year.* Consider an ideal gas of temperature T, composed of atoms with mass m and number density n, confined to a box.

   A) Determine the number of particles striking a unit area of wall per unit time, using the Maxwell speed distribution

$$P(v)\,dv = 4\pi v^2\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-mv^2/2kT\right)\,dv \quad .$$

   You may need to use some of the integrals given at the end of the exam.

   B) Consider an oven containing cadmium vapor at a pressure of 2.28 Pa and at a temperature of 550K. In one wall of the oven there is a slit with a width of 10^{-5}m and a length of 10^{-2}m. On the other side of this wall is a very high vacuum. If one assumes that all the atoms arriving at the slit pass through, find the atomic beam current, in atoms per second. The atomic weight of cadmium is 112. Still missing this. Take one from standard book.

99_OL04
3:20pm

University of Wyoming
Department of Physics and Astronomy
Written Preliminary Examination – Part I. Friday, January 15, 1999
9:00 am – 2:00 PM
Instructions for Part I

1. Do seven (7) problems as instructed on the following pages.
2. Answer each problem on a separate set of paper.
3. Enter the identification number assigned to you on the upper right hand corner of each answer page. Do not write your name on these pages.
4. Some integrals, numerical constants, and mathematical expressions which might be useful are provided on the data pages at the end of the examination.
5. Each problem will contribute equally to your grade on this part of the examination.
6. Partial credit will be given for orderly progress towards a correct solution of a problem; however no credit will be given for an illogical, disorganized presentation or for irrelevant material

99_12-08a 4:55pm

University of Wyoming
Department of Physics and Astronomy
Written Preliminary Examination – Part II.
For ASTRONOMY students.
December 1999
5 hours
Instructions for Part II

1. Do six (6) problems as instructed on the following pages.
2. Answer each problem on a separate set of paper.
3. Enter the identification number assigned to you on the upper right hand corner of each answer page. Do not write your name on these pages.
4. Some integrals, numerical constants, and mathematical expressions which might be useful are provided on the data pages at the end of the examination.
5. Each problem will contribute equally to your grade on this part of the examination.
6. Partial credit will be given for orderly progress towards a correct solution of a problem; however no credit will be given for an illogical, disorganized presentation or for irrelevant material

Part II: Section A

Do each of the four problems in Section A.

1. **Mathematical Methods.** Obtain a solution of the following differential equation

$$\frac{d^2x(t)}{dt^2} = \alpha x^2(t)$$

good to the fourth power in the constant α. That is, use terms up to and **including** α^4. Use the initial conditions

$$x(t)\,|_{t=0} = 1$$

$$\frac{dx}{dt}\,|_{t=0} = 0 \quad .$$

Obtain the region of t (for $t > 0$) over which your solution is expected to be valid.

2. **Classical Mechanics.** Consider a hoop of mass M and radius r rolling without slipping down an inclined plane. The plane is inclined at angle ϕ. Assume at $t = 0$ the hoop is at the top of the inclined plane and is released at rest.

   A) Write down the Lagrangian for the system. Indicate clearly in a diagram your choice of coordinates.

   B) Obtain the Hamiltonian for the system.

   C) Write the Hamilton-Jacobi equation.

   D) Solve the Hamilton-Jacobi equation and obtain the velocity of the hoop when it reaches the bottom of the inclined plane. Express this answer in terms of the angle of the incline ϕ and the distance L which the hoop has rolled along the diagonal.

3. **Quantum Mechanics.**

   A) Suppose that one has an atom with only two states $|+\rangle$ and $|-\rangle$, with energies E_+ and E_-. What is the Hamiltonian H_o of the two level atom, in the $(\,|+\rangle\,,\,|-\rangle\,)$ representation?

   B) Suppose that the expectation value of the electric dipole moment (ez) is zero for both $|+\rangle$ and $|-\rangle$, but that $\langle +|ez|-\rangle = \mu$ is real and nonzero. Write down the Hamiltonian of the system in the presence of an electric field $\bar{E}\cos(\omega t)$ in the z direction, again using the above representation.

   C) For part (B) write the time dependent Schrödinger equation. Solve the time dependent Schrödinger equation for the case where the wave function is $|+\rangle$ at $t = 0$ and a constant electric field $(\omega = 0)$ is switched on at $t = 0$.

4. **Electricity and Magnetism.** Consider a medium with nonzero conductivity σ ($\vec{J} = \sigma\vec{E}$ gives the current density) and no net charge ($\rho = 0$).

A) Write the set of Maxwell's equations appropriate for this medium.

B) Derive the wave equation for $\vec{E}$ (or $\vec{B}$) in this medium,

$$[\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V}] \quad .$$

C) Consider a monochromatic wave moving in the $+x$ direction with E_y (or E_z or B_y or B_z) given by

$$E_y = \psi = \psi_0 e^{i(kx-\omega t)} \quad .$$

Show that this wave has an amplitude which decreases exponentially; find the attenuation length (skin depth).

D) For sea water ($\sigma \approx 5$ mho/m, or 4.5×10^{10} s^{-1} in cgs units), and using radio waves of long wavelength $\omega = 5 \times 10^5\text{s}^{-1}$, calculate the attenuation length. (Why is it hard to communicate with submerged submarines?) You can take for sea water $\epsilon = 1$, $\mu = 1$.

Part II: Section B

Do any two of the four problems in Section B.

5. Assume the Sun has a temperature structure which results in it being composed of a uniformly-dense ideal gas with mean molecular weight $\mu = 1/2$. Integrate the stellar structure equations to find the core temperature, T_c, in terms of the mass M, and the radius R. What value of T_c results from this estimate?

6. **Astronomical Techniques.** The read noise, gain, and saturation level of a CCD or IR array camera can be determined in the laboratory by obtaining a series of exposures of a flat field light source, even if the absolute intensity of that source is not known. One uses exposures of different length, and relies upon the statistical properties of photon noise. Describe the specific experimental procedure in terms of what type of exposures one must obtain. Assume that the array has the usual imperfections such as slightly non-uniform flat field response, some dark current, etc.

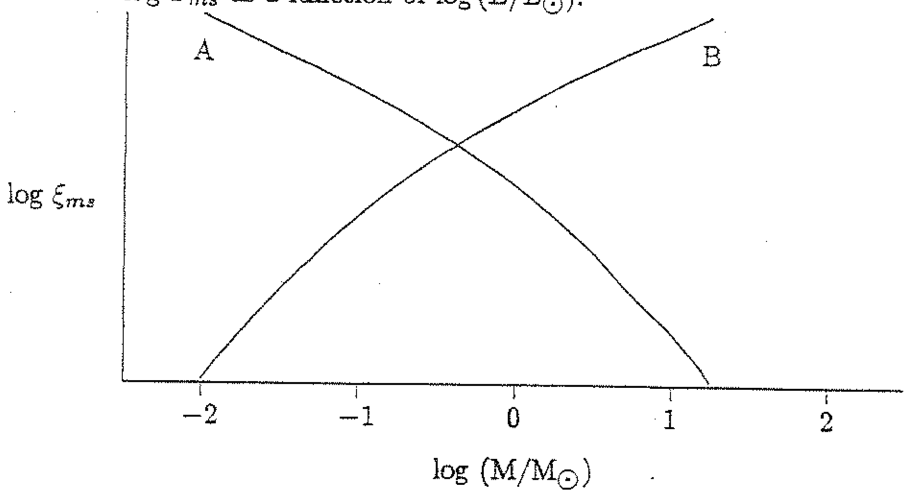
   Derive an equation for the gain g in DN/e^- or ADU/e^- (i.e. Data Number or Analog-to-Digital-Converter Units per electron), in terms of the measured properties of the images. Also specify the read noise in terms of both DN and electrons. Describe qualitatively how you determine the saturation level, both in terms of DN and electrons. You may find it useful to make a sketch of a plot involving two important measured quantities and label your derived quantities on that plot. Be sure to define all symbols you use and label the axes of your plot, and give units for those axes.

7. Galaxies and Cosmology
   - A) Write down the Friedman Robinson Walker metric, that is, the metric for a homogeneous isotropic universe.
   - B) Derive the relation between the cosmological redshift and the cosmological expansion parameter $R(t)$.
   - C) Derive the angular size vs. redshift for an object of constant size, assuming $\Omega = 1$, so that $R(t) \propto t^{2/3}$. Leave your answer in terms of H_o and z. Hint: You will need to derive τ vs. t.

8. **ISM/Stellar Populations**

A) Shown below are two possible initial mass functions. Which is the one which approximates real observations of the IMF? State in words what this function tells about the relative abundance of different type main sequence stars.

B) Based on the correct IMF, draw the corresponding main sequence luminosity function $\log \Psi_{ms}$ as a function of $\log (L/L_{\odot})$.



Integrals

(1) $$\int_0^\infty \frac{dy}{1+y^2} = \frac{\pi}{2}\ .$$

(2) $$\int_0^\infty \frac{\sin(qr)}{r}\,dr = \frac{\pi}{2}$$

(3) $$\int_0^\pi 2\pi \frac{\sin\theta}{\sin^2\frac{\theta}{2}}\,d\theta = \infty$$

(4) $$\int_0^\infty r\exp\left((\pm iq - \frac{1}{R})r\right)dr = \frac{1}{\left((\pm iq - \frac{1}{R})\right)^2}\ .$$

(5) $$\int_0^\pi \frac{2\pi\sin\theta}{\left(1+4k^2R^2\sin^2\frac{\theta}{2}\right)^4}\,d\theta = \frac{\pi}{3}\cdot\frac{1}{k^2R^2}\left[1-\frac{1}{\left(1+4k^2R^2\right)^3}\right].$$

(6) $$\int_0^\pi \frac{2\pi\sin\theta}{\left(1+4k^2R^2\sin^2\frac{\theta}{2}\right)^2}\,d\theta = \frac{\pi}{\left(1+4k^2R^2\right)}$$

Mathematical Relations

$$\int_0^\infty e^{-ax^2} x^{2n}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\frac{\Gamma(n+\frac{1}{2})}{a^n\,\Gamma(\frac{1}{2})}, \quad (a>0; \quad n \text{ an integer} \geq 0)$$

$$\int_0^\infty e^{-ax^2} x^{2n+1}\, dx = \frac{n!}{2a^{n+1}}, \quad (a>0; \quad n \text{ an integer} \geq 0)$$

$$\int_0^\infty e^{-ax} x^n\, dx = n!/a^{n+1}, \quad (a>0; \quad n \text{ an integer} \geq 0)$$

$$n! \approx \sqrt{2\pi} \quad n^{n+\frac{1}{2}} e^{-n}, \quad \text{for } n >>> 1$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$\int e^{ax} \sin(bx)\, dx = (a^2+b^2)^{-1}\, e^{ax}(a\,\sin(bx) - b\,\cos(bx))$$

$$\int x\,\sin^2(ax)\, dx = \frac{x^2}{4} - \frac{x\,\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int \sin^3(x)\, dx = -\frac{1}{3}\cos(x)\,(\sin^2 x + 2)$$

$$\int \sin^4(x)\, dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}$$

$$\int_0^\infty e^{-ax^2} \cos(bx)\, dx = \frac{1}{2}(\pi/a)^{\frac{1}{2}}\, e^{-b^2/4a}$$

Cylindrical Coordinates

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \vec{V} = \hat{r}(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}) + \hat{\theta}(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}) + \hat{z}\frac{1}{r}(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta})$$

$$\nabla \Phi = \hat{r}\ \frac{\partial \Phi}{\partial r} + \hat{\theta}\ \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{z}\ \frac{\partial \Phi}{\partial z}$$

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \Phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Spherical Coordinates

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\ V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

$$\nabla \times \vec{V} = \hat{r}\ \frac{1}{r\sin\theta}(\frac{\partial}{\partial \theta}(\sin\theta\ V_\phi) - \frac{\partial V_\theta}{\partial \phi} + \hat{\theta}\ (\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rV_\phi)) + \hat{\phi}\ \frac{1}{r}(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta})$$

$$\nabla \Phi = \hat{r}\ \frac{\partial \Phi}{\partial r} + \hat{\theta}\ \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{\phi}\ \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}$$

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \Phi}{\partial r}) + \frac{1}{r^2}(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial \Phi}{\partial \theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2})$$

PHYSICAL CONSTANTS

$$1\ \mathring{A} = 10^{-8}\text{cm}$$

$$1\ \text{fermi} = 10^{-13}\text{cm}$$

$$1\ \text{parsec} = 3.087 \times 10^{18}\text{cm}$$

$$G = 6.67 \times 10^{-11}\text{nt} - \text{m}^2/\text{kg}^2$$

$$k_B = 1.38 \times 10^{-23}\ \text{joule/K} = 1.38 \times 10^{-16}\ \text{erg/K}$$

$$N_A = 6.02 \times 10^{23}\ \text{particles/mole}$$

$$e = 1.60 \times 10^{-19}\ \text{coulomb}$$

$$c = 2.998 \times 10^{10}\text{cm/sec}$$

$$\sigma = 5.67 \times 10^{-8}\ \text{joule/(deg}^4 - \text{m}^2 - \text{sec)}$$

$$1eV = 1.60 \times 10^{-19}\ \text{joule}$$

$$h = 6.626 \times 10^{-34}\ \text{joule} - \text{sec} = 6.626\ 10^{-27}\text{erg} - \text{sec}$$

$$\hbar c = 197MeV - \text{fm} = 197\ eV - \text{nm}$$

$$m_e = 0.511MeV/c^2 = 9.11 \times 10^{-31}\text{kg}$$

$$m_\pi = 139.6MeV/c^2$$

$$m_\mu = 105.7MeV/c^2$$

$$m_{proton} = 938MeV/c^2 = 1.67 \times 10^{-24}\text{gm}$$

$$\tau(\pi) = 2.60 \times 10^{-8}\text{sec}$$

University of Wyoming
Department of Physics and Astronomy
Written Preliminary Examination - Part II.
Saturday, January 24, 1998
9:00 am - 2:00 pm
Instructions for Part II
PHYSICS

1. Do six (6) problems as instructed on the following pages.
2. Answer each problem on a separate set of paper.
3. Enter the identification number assigned to you on the upper right hand corner of each answer page. Do not write your name on these pages.
4. Some integrals, numerical constants, and mathematical expressions which might be useful are provided on the data pages at the end of the examination.
5. Each problem will contribute equally to your grade on this part of the examination.
6. Partial credit will be given for orderly progress towards a correct solution of a problem; however no credit will be given for an illogical, disorganized presentation or for irrelevant material

Part II: Section A

Do <u>each</u> of the four problems in Section A.

1. **Mathematical Methods.** Consider the differential equation

$$\frac{d^2 f(t)}{dt^2} = 1 + \eta \left(\frac{df(t)}{dt} \right)$$

with

$$f(0) = 1 \quad \text{and} \quad \left(\frac{df}{dt} \right)_{t=0} = 0 \quad .$$

Solve it in two different ways:

A) First find the solution in the form

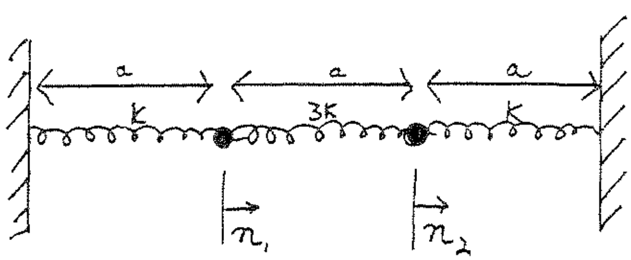
$$f(t) = f_0(t) + \eta f_1(t)$$

assuming η is small.

B) Next find the exact solution for $f(t)$.

C) Finally, show that the exact solution, in the limit $\eta \to 0$, gives the same result as in part A.

2. **Classical Mechanics.** Two particles each of mass m move in one dimension at the junction of the massless springs shown. The middle spring has force constant $3K$ and the end springs have force constant K. All springs have relaxed length a. Write the Lagrangian and find the eigenfrequencies and normal modes of the system.



3. **Electricity & Magnetism** In polar coordinates (r, ϕ) the two-dimensional Laplace's Equation for the electrostatic potential $V(r, \phi)$ has the general solution for $0 \leq r \leq \infty$ and $0 \leq \phi \leq 2\pi$

$$V(r, \phi) = (A_o + B_0\ \phi)(C_0 + D_0 \ln r) + \sum_{l=1}^{\infty} \left[A_l r^l + B_l r^{-l}\right]\left[C_l \cos l\phi\ + D_l \sin l\phi\ \right] \quad ,$$

where A_0, B_0, ... and A_l, B_l, ... are constants determined by the particular application

Consider the two-dimensional region interior to the lines $\phi = 0$, $\phi = \pi/2$, and $r = R$, which is filled with a dielectric of permittivity ϵ_1 for $0 \leq r \leq a$ and with a dielectric of permittivity ϵ_2 for $a \leq r \leq R$. The bounding lines of the overall region are all conductors with

$$(1) \quad V(r, 0) = 0, \quad (2) \quad V(r, \pi/2) = 0, \quad \text{and } (3) \quad V(R, \phi) = V_0 \quad .$$

There is no free charge density anywhere except possibly on the conductors.

A) Use boundary conditions (1) and (2) and any other appropriate physics to show that we may write

$$V(r \leq a, \phi) = \sum_{l=1}^{\infty} a_{2l}\ r^{2l} \sin 2l\phi$$

$$V(a \leq r \leq R, \phi) = \sum_{l=1}^{\infty} \left[b_{2l}\ r^{2l} + c_{2l}\ r^{-2l}\right] \sin 2l\phi$$

where a_{2l}, b_{2l}, and c_{2l} are constants yet to be determined.

B) Use the appropriate boundary conditions at $r = a$ to determine two relations among the three constants a_{2l}, b_{2l}, and c_{2l}.

C) Show how to determine a third relation among a_{2l}, b_{2l}, and c_{2l} but do *not* carry out any integrations.

D) Assuming the a_{2l}, b_{2l}, c_{2l} are now known, *show how to* calculate all surface charge distributions, but *do not carry out the actual calculations.*

4. **Quantum Mechanics** The normalized wavefunction for the ground state of a hydrogen-like atom with nuclear charge Ze has the form

$$u(r) = Ae^{-\alpha r} \quad ,$$

where A and α are constants and r is the relative coordinate between the electron of mass m and nucleus. By deriving explicit expressions for the integrals

$$I_n \equiv \int_0^{\infty} r^n e^{-\alpha r}\ dr \quad ,$$

which readily follow from first evaluation I_0, study this ground state in the following steps.

A) Determine A and α in terms of fundamental physical constants – e.g., the first Bohr radius, $a_0 = \hbar^2/me^2$.

B) Similarly, find the ground state energy.

C) Compute the expectation values of the potential and kinetic energies.

D) Find the expectation value of r.

E) Find the most probable value of r.

It may be useful to recall that in spherical coordinates the radial component of ∇^2 is $\partial^2/\partial r^2 + (2/r)\partial/\partial r$.

Part II: Section B

Do any two of the three problems in Section B.

5. **Statistical Mechanics** A system is characterized by the two energy levels

$$-\epsilon \quad \text{and} \quad \epsilon \ .$$

It is maintained in a state such that the average energy $\bar{E}$ of the system is found to be

$$\bar{E}/k_B = 1^\circ \text{ Kelvin}$$

where k_B is the Boltzmann constant. Although our classical understanding of temperature may be of marginal use for such a two level system, we can still define temperature in terms of partition functions or basic definitions of entropy and temperature.

A) Find the temperature of this system, when it has the $\bar{E}$ given above.

B) Show whether that temperature is negative or positive.

C) Conventional wisdom says that temperature is positive. Reconcile this with your result in part b.

6. **Advanced Quantum Mechanics.**

A) State clearly the assumptions and physical principles that led to the development of the Dirac equation

$$i\hbar\frac{\partial\psi}{\partial t} = H_D\psi$$

$$H_D = c\vec{\alpha}\cdot(\vec{p} - \frac{e\vec{A}}{c}) + \beta(mc^2 + e\Phi)$$

where α_x, α_y, and α_z are Dirac matrices.

B) Solve for the eigenfunctions of the free particle Dirac Hamiltonian for the case when the particle is moving in one dimension. Do not bother normalizing the eigenfunctions.

C) What would you expect for the form of the solutions in the nonrelativistic limit?

7. **Electricity and Magnetism – Waves**

Consider a rectangular waveguide of width a in x and b in y, with a wave propagating in the z direction. Assume the waveguide contains a vacuum and the walls are infinitely conducting. Also assume $a > b$. There will be some lowest frequency TE mode, denoted TE_{10} which can propogate through the system. Find the frequency of this mode. In addition, using Maxwell's equations, the wave equation derived from them, and the appropriate boundary conditions, find the $\vec{E}$ and $\vec{H}$ fields, as functions of (x, y, z, t), for this mode.

Mathematical Relations

$$\int_0^\infty e^{-ax^2} x^{2n}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\frac{\Gamma(n+\frac{1}{2})}{a^n\Gamma(\frac{1}{2})}, \quad (a > 0; \quad n \text{ an integer} \geq 0)$$

$$\int_0^\infty e^{-ax^2} x^{2n+1}\, dx = \frac{n!}{2a^{n+1}}, \quad (a > 0; \quad n \text{ an integer} \geq 0)$$

$$\int_0^\infty e^{-ax} x^n\, dx = n!/a^{n+1}, \quad (a > 0; \quad n \text{ an integer} \geq 0)$$

$$n! \approx \sqrt{2\pi} \quad n^{n+\frac{1}{2}} e^{-n}, \quad \text{for } n >>> 1$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$\int e^{ax} \sin(bx)\, dx = (a^2+b^2)^{-1}\; e^{ax}(a\; \sin(bx) - b\; \cos(bx))$$

$$\int x\; \sin^2(ax)\, dx = \frac{x^2}{4} - \frac{x\; \sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int \sin^3(x)\, dx = -\frac{1}{3}\cos(x)\,(\sin^2 x + 2)$$

$$\int \sin^4(x)\, dx = \frac{3x}{8} - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32}$$

$$\int_0^\infty e^{-ax^2} \cos(bx)\, dx = \frac{1}{2}(\pi/a)^{\frac{1}{2}}\; e^{-b^2/4a}$$

Cylindrical Coordinates

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \vec{V} = \hat{r}(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}) + \hat{\theta}(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}) + \hat{z}\frac{1}{r}(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta})$$

$$\nabla \Phi = \hat{r}\ \frac{\partial \Phi}{\partial r} + \hat{\theta}\ \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{z}\ \frac{\partial \Phi}{\partial z}$$

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \Phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

Spherical Coordinates

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\ V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

$$\nabla \times \vec{V} = \hat{r}\ \frac{1}{r\sin\theta}(\frac{\partial}{\partial \theta}(\sin\theta\ V_\phi) - \frac{\partial V_\theta}{\partial \phi} + \hat{\theta}\ (\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rV_\phi)) + \hat{\phi}\ \frac{1}{r}(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta})$$

$$\nabla \Phi = \hat{r}\ \frac{\partial \Phi}{\partial r} + \hat{\theta}\ \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{\phi}\ \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}$$

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \Phi}{\partial r}) + \frac{1}{r^2}(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial \Phi}{\partial \theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2})$$

PHYSICAL CONSTANTS

$$1\ \mathring{A} = 10^{-8}\text{cm}$$

$$1\ \text{fermi} = 10^{-13}\text{cm}$$

$$1\ \text{parsec} = 3.087 \times 10^{18}\text{cm}$$

$$G = 6.67 \times 10^{-11}\text{nt} - \text{m}^2/\text{kg}^2$$

$$k_B = 1.38 \times 10^{-23}\ \text{joule/K} = 1.38 \times 10^{-16}\ \text{erg/K}$$

$$N_A = 6.02 \times 10^{23}\ \text{particles/mole}$$

$$e = 1.60 \times 10^{-19}\ \text{coulomb}$$

$$c = 2.998 \times 10^{10}\text{cm/sec}$$

$$\sigma = 5.67 \times 10^{-8}\ \text{joule/}(\text{deg}^4 - \text{m}^2 - \text{sec})$$

$$1eV = 1.60 \times 10^{-19}\ \text{joule}$$

$$h = 6.626 \times 10^{-34}\ \text{joule} - \text{sec} = 6.626\ 10^{-27}\text{erg} - \text{sec}$$

$$\hbar c = 197MeV - \text{fm} = 197\ eV - \text{nm}$$

$$m_e = 0.511MeV/c^2 = 9.11 \times 10^{-31}\text{kg}$$

$$m_\pi = 139.6MeV/c^2$$

$$m_\mu = 105.7MeV/c^2$$

$$m_{proton} = 938MeV/c^2 = 1.67 \times 10^{-24}\text{gm}$$

$$\tau(\pi) = 2.60 \times 10^{-8}\text{sec}$$