Ordinary Differential Equations I: Basic Methods

You have a differential equation to solve, for example

$\Sigma \boldsymbol{F} = m\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t = m\, \mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$ This is an ordinary differential equation of second order

<u>Numerical Integration</u>

If the force does not depend on the velocity, then

$\boldsymbol{v} = \int \boldsymbol{F}/m\, \mathrm{d}t$

$\boldsymbol{r} = \int \boldsymbol{v}\, \mathrm{d}t$

Sometimes the force contains derivatives, sometimes the integral of $\boldsymbol{F}/m$ is simply not analytic, and sometimes the the force values come from a measurement and/or a data table: you must “numerically integrate.”

<u>Numerical Differentiation</u>

Alternatively, you may have the data table for the speed of a rocket and wish to compute the acceleration (and hence the force applied to it).

$\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$

Taylor series approximation for the forward derivative:

$v(t+\delta t) = v(t) + v'(t)\delta t + v''(t)\, \delta t^2/2 + \ldots$

or

$a(t) = v'(t) = [v(t+\delta t) - v(t)]/\delta t - v''(t)\, ½\, \delta t + \ldots$

Euler Method

The equations of motion are solved iteratively through the use of

$$a(t) = [v(t+\delta t) - v(t)]/\delta t + O(\delta t)$$
$$v(t) = [r(t+\delta t) - r(t)]/\delta t + O(\delta t)$$

or

$$v(t+\delta t) = v(t) + \delta t\, a(t) + O(\delta t^2) \qquad \text{I}$$
$$r(t+\delta t) = r(t) + \delta t\, v(t) + O(\delta t^2) \qquad \text{II}$$

Program outline:

1. Specify initial position and velocity $r(t_o)$ and $v(t_o)$
2. Find the acceleration given the ________ position and velocity
3. Use Equations I and II above to compute the _______ position and velocity $r(t+\delta t)$ and $v(t+\delta t)$
4. Go back to Step 2 until finished

Round-off error and truncation error

See p. 28 & p. 39. Is there a difference in round-off and truncation error, or are they just jargon for the same thing?

Q: Why do you think *v* in the above first equation pair has a larger truncation error than the second equation pair?

A:

In the first half of this chapter we'll analyze the flight path of a baseball, with and without air resistance. Including air resistance makes this a more difficult problem; we'll use computational techniques to derive a numerical approach to the solution.

$$\boldsymbol{F} = m\boldsymbol{g} + \boldsymbol{F}_a(\boldsymbol{v})$$

where $\boldsymbol{F}_a(\boldsymbol{v}) = -0.5 C_d \rho A |\boldsymbol{v}| \boldsymbol{v}$

In which direction does the drag force act?

The coefficient of drag C_d is______ for small velocities and ______ for large velocities.

For a well hit baseball traveling at ~40 m/s, the drag and gravitational forces are equal in magnitude.

a. What does this imply for a “sky high” pop-up to the catcher?

b. What does this imply for a “line drive” basehit to center field?

Figure 2.2 shows the effects of “truncation error” for numerically computing the baseball's flight path. Figure 2.3 shows the effects of adding air resistance for a well-hit ball. To paraphrase page 44 discussion: the effects of air drag are quite obvious for a ball that is hit hard (i.e., fast), and not for a poorly-hit ball (i.e., slow). But wait a minute: I thought we agreed that the drag coefficient was large for small velocities and vice-versa? Why do the effects of air resistance only become obvious for fast baseballs?

Always try to check your numerical answer against a 'close' problem with a known analytic answer.

Quick review:

- truncation error vs round-off error (dropping higher order terms vs numerical storage limitations of cptr)
- time steps can be gauged conceptually, and through truncation error analysis (but not too small or you will encounter round-off error)
- if on page 2 of these notes we say the truncation error is $+O(\delta t^2)$, then why are the data suffering from truncation error on Figure 2.2 on p. 44 *greater* than those for the theoretical curve? i.e., shouldn't dropping a term of order $+O(\delta t^2)$ imply *smaller* displacements in x and y?

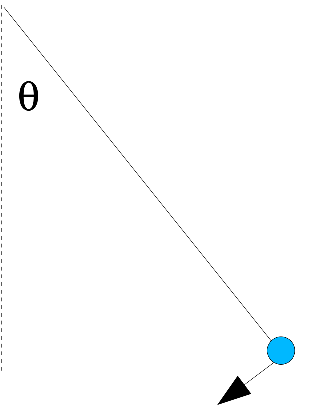
Simple Pendulum

The simple pendulum problem can be solved in different ways depending on the amplitude of oscillation.

Small oscillations

$d^2\theta/dt^2 = -g/L \sin\theta \sim -g/L\ \theta$

$\omega = (g/L)^{1/2}$



This is a canonical problem in physics, e.g., Hooke's Law

$d^2x/dt^2 = -k/m\ x \quad \rightarrow \quad \omega = (k/m)^{1/2}$

The solution is straightforward: $\theta = \theta_o \cos(\omega t)$ *does this satisfy all scenarios?*

Large oscillations

For angles that don't satisfy the small angle approximation $\sin\theta \sim \theta$, we can use an energy analysis

E = constant

$E(t) = \frac{1}{2}\, m\, L^2\, \omega(t)^2 - mgL \cos\theta(t)$

$E(t=0) = ?$

$\omega = d\theta/dt = (2gL^{-1}[\cos\theta - \cos\theta_o])^{1/2}$

$\rightarrow \quad dt = d\theta/(2gL^{-1}[\cos\theta - \cos\theta_o])^{1/2}$

using trig. identities and change of variables yields

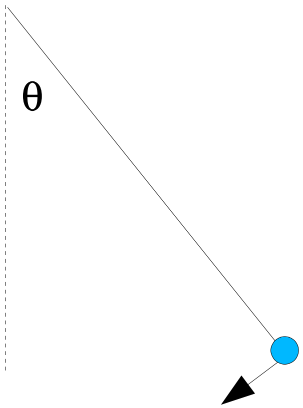
$T = 2\pi/\omega = \int dt = 4\,(L/g)^{1/2}\, K(\sin\frac{1}{2}\theta_o)$

for small angles, $T = 2\pi\,(L/g)^{1/2}\,(1 + 1/16\ \theta^2_o + \ldots)$ *is this consistent with the above solution?*

The numerical solution using the Verlet method is

$$\theta_{n+1} = 2\theta_n - \theta_{n-1} + \delta t^2 \alpha_n$$

In our Just-in-Time question, we saw that the Verlet and leap-frog methods conserved energy, whereas the Euler method did not. How were you able to answer this using the plots on pp. 53-55?



One question I didn't ask: Why do the Verlet and leap-frog methods conserve energy, whereas the Euler (and Euler-Cromer) method does not?

Q: If $v=L\omega$, then ω must change since v changes. But doesn't ω also equal $2\pi/T$, where T is constant?
A:

Slide 1

In gap in Slide 1, show graphs of ***F*** vs *t* and ***v*** vs *t*; make rectangles of width δt; use midpoint rule to estimate height of rectangle (see my notes on yellow engineering paper).

Point them to Eqns 2.7 and 2.8

$a(t) = v'(t) = [v(t+\delta t) - v(t)]/\delta t - v''(t)\ \tfrac{1}{2}\ \delta t\ + ...$ ie. a is slope of *v* vs *t*; leftovers 'converge'; the approximation improves as $\delta t \to 0$

Q: what happened to the δt^2 in the above equation?

A: we divided through by δt

Slide 2

Explain last term in Eqn 2.10 is *truncation* error.

2. Find the acceleration given the *current* position and velocity

3. Use Equations I and II above to compute the *next* position and velocity $r(t+\delta t)$ and $v(t+\delta t)$

What do we mean by “compute *a* using current *r* and *v*”? Doesn't the first equation on Slide 2 show that we need to use future *v* and current *v*? No, show class that we compute *a* from dividing Equation 2.2 on p. 37 by mass. Ask class this question, in fact.

Round-off error and truncation error

see Fig 1.3 on p. 29

Q: Why does it go haywire below e-8?

A: round-off error

Q: Is there a difference between round-off and truncation, or is it just jargon for the same thing? Both pp. 28-29 and p. 39 are dealing with approximations to a derivative.

A: Round-off error comes from limited ability of computer to compute small differences, whereas truncation is ignoring higher-order terms.

Q: Why do you think the above first equation pair has a smaller truncation error than the second equation pair?

A: It's better to estimate velocity at time *t* with the knowledge at the same time step, versus needing to know information from two different time steps.

Slide 3

Free-body diagram

Do demo with a coffee filter and a crumpled coffee filter – motivate that it should depend on area A. Show astronaut video. Do vacuum tube demo.

Ask class about the terminal velocity of a human sky diver (60 m/s or ~120 mph). Derive it. $0.5C_d\rho A|\mathbf{v}|\mathbf{v} = mg$ --> v=sqrt(mg/A0.5$C_d\rho$) ~ sqrt(800/0.125ρ)

Work on board the first few iterations for a dropped baseball using the Euler method.

Edit code balle.f to show results; use tau=0.01 to recover handwritten table values.

See yellow engineering paper: y0=100; v0=0

Show how the numbers change with larger and smaller step sizes; do it with and without air resistance.

w/resistance final values: t_{tot}=4.99 a_{final}=-2.822 v_{final}=-33.52603

w/out resistance: t_{tot}=4.52 a_{final}=-9.810 v_{final}=-44.43930 (shorter and faster)

Is there anything goofy about these values? Yes: y and a don't change from Step 0 to Step 1, though v does change. Fact of life in numerical approximations. Smaller δt would help, but that increases computing time and round off error may increase. Also, y_{final}<0.

In which direction does the drag force act? Upwards (or more precisely, opposite the velocity)

The coefficient of drag C_d is__large____ for small velocities and ___small___ for large velocities. (constant at medium v; C_d inversely proportional to v for small v)

For a baseball traveling at ~40 m/s, the drag and gravitational forces are equal in magnitude.

a. What does this imply for a "sky high" pop-up to the catcher? Quickly decelerates, much more so than without air resistance (cf on the moon)

b. What does this imply for a "line drive" basehit to center field? Decelerates in x as much as it does in y -> parabolic path no longer holds

But wait a minute: I thought we agreed that the drag coefficient was large for small velocities and vice-versa? But then why do we always think that the effects of air resistance only become obvious for fast baseballs? It's because the drag *force* is a function of both C_d and velocity (squared), and thus the drag force is quite dramatic if v is large. Note also that at 15 m/s, the book indicates that C_d is approximately constant (only increases with decreasing v for v<2 m/s).

Read 1.22 of Flying Circus of Physics (2nd edition). He's wrong: the final velocity of a baseball dropped in air from a height of 213 m is about 137 km/h (38.3 m/s). Either he was thinking mph when it was really km/h, or he used no air drag (64.7 m/s or 233 km/h).

Also, google Felix Baumgarnter skydive: 36600m, 330s until parachute deployment @ 1550m. Check w/balle.f: rho_air is smaller, so he went faster.

Slide 4

conceptually gauges --> see p. 42

if on page 2 of these notes we say the truncation error is $+O(\delta t^2)$, then why are the data suffering from truncation error on Figure 2.2 *greater* than those for the theoretical curve? ie. shouldn't dropping $+O(\delta t^2)$ imply smaller displacement?

It's because we are dropping the acceleration terms for x and y (eq 2.24), and both of these values are in the negative direction – dropping a negative means it's artificially positive.

Slide 5

Draw the free-body diagram and derive $d^2\theta/dt^2 = -g/L \sin\theta \sim -g/L\ \theta$

$m d^2x/dt^2 = -\ mg \sin(\theta)$ and $x=L\theta$

Then plug in $\theta = \theta_o \cos(\omega t)$ and show it works

Q: Does this satisfy all scenarios?

A: No, only for small angles

$E_f = ½\ mv^2 + mgh$, with $v=L\omega$ and $h=-L\cos(\theta)$ leads to $E_f = ½\ m\ L^2\ \omega^2 - mgL \cos\theta$

Group:

Q: E_i=? To put another way, Why is $h=-L\cos(\theta)$? Or: Why is the total energy negative? (since ω=0 when released -> $E_i=-mgL\cos\theta_{released}$)

A: the textbook author has decided to define zero potential as a 90 degree angle away from vertically hanging down

You get the equation for ω from setting $E_o=E_{final}$.

$T=\int dt = \int d\theta/(2gL^{-1}[\cos\theta-\cos\theta_o])^{½}$

$T/4 \quad = \sqrt{L/2g}\ \int d\theta/[\cos\theta-\cos\theta_o]^{½}$ from 0 to θ_o

$= \sqrt{L/2g}\ \int d\theta/[2\sin^2(\theta_o/2)-2\sin^2(\theta/2)]^{½}$ from 0 to θ_o since $\cos 2\theta=1-2\sin^2\theta$

which implies

$T \quad = 2\sqrt{L/g}\ \int d\theta/[\sin^2(\theta_o/2)-\sin^2(\theta/2)]^{½}$ from 0 to θ_o

$= 2\sqrt{L/g}\ 1/\sin(\theta_o/2)\ \int d\theta/[1-\sin^2(\theta/2)/\sin^2(\theta_o/2)]^{½}$ from 0 to θ_o

let $\sin z=\sin(\theta/2)/\sin(\theta_o/2)$ -> $\cos z\ dz = ½ \cos(\theta/2)\ d\theta / \sin(\theta_o/2)$

$= 2\sqrt{L/g}\ 1/\sin(\theta_o/2)\ \int 2\cos z\ dz\ \sin(\theta_o/2) / \cos(\theta/2) / [1-\sin^2(z)]^{½}$ from 0 to $\pi/2$

$= 4\sqrt{L/g}\ \int dz\ 1 / \cos(\theta/2)$ from 0 to $\pi/2$

$= 4\sqrt{L/g}\ \int dz\ 1 / [1-\sin^2(\theta/2)]^{½}$ from 0 to $\pi/2$

$= 4\sqrt{L/g}\ \int dz\ 1 / [1-\sin^2(\theta_o/2)\sin^2(z)]^{½}$ from 0 to $\pi/2$

$= 4\sqrt{L/g}\ K(\sin[\theta_o/2])$

Slide 6

Q: In our Just-in-Time question, we saw that the Verlet and leap-frog methods conserved energy, whereas the Euler method did not. How were you able to answer this using the plots on pp. 53-55?

A: Energy scales as $mgL \cos\theta_m$ so just look at the maximum angle

Why is Verlet better than Leap-Frog, which in turn is better than Euler? The truncation errors differ.

Verlet	$r_{n+1} = 2r_n - r_{n-1} + \tau^2 a_n + O(\tau^4)$	(2.59)
Leap-Frog	$r_{n+2} = r_n + 2\tau v_{n+1} + O(\tau^3)$	(2.51)
Euler-Cromer	$r_{n+1} = r_n + \tau v_{n+1} + O(\tau^2)$	(2.21)
Euler	$r_{n+1} = r_n + \tau v_n + O(\tau^2)$	(2.19)

Q: If $v=L\omega$, then ω must change since v must change. But doesn't ω also equal $2\pi/T$, where T is a constant?

A: I'm mixing instantaneous versus time-averaged quantities. In the first case ω is an instantaneous quantity, whereas in the latter case it's averaged over (many) cycles.

Homework 1 Recap:

#23: you can combine terms to remove the minus sign

$$a_{head} - a_{feet} = v^2/r - v^2/r = (2\pi(R+h)\cos\phi)^2/([R+h]\cos\phi T^2) - (2\pi(R)\cos\phi)^2/(R\cos\phi T^2) = 4\pi^2 h\cos\phi/T^2$$

#26: for negative x the series is alternating plus and minus; terms need to cancel each other out, leaving small remainder that is susceptible to round-off error

Terminal velocity appendix

http://www.reddit.com/r/askscience/comments/1193gy/if_the_terminal_velocity_for_humans_is_around_125/

See terminal.speed.txt file