Partial Differential Equations II

While Chapter 6 introduced the three categories of PDEs (hyperbolic, parabolic, elliptic), it only focused on solving the parabolic PDE (in the guise of the diffusion equation). Chapter 7 concentrates on solving the hyperbolic form of the PDE, and Chapter 8 the elliptic version. We will only cover Chapters 6 & 7 this semester, so that we will have time to cover the fun and useful topic of Stochastic Methods (Chapter 11). *Adapted from Numerical Recipes*

The hyperbolic wave equation (speed v)

$$\partial^2 u/\partial t^2 = v^2 \, \partial^2 u/\partial x^2$$

can be rewritten as two first-order differential equations

$$\partial r/\partial t = v\partial s/\partial x \qquad \partial s/\partial t = v\partial r/\partial x \qquad \text{where} \qquad r \equiv v\partial u/\partial x \qquad \& \qquad s \equiv \partial u/\partial t$$

Defining **u** as a column vector with components r and s yields the linear matrix relation

$$\mathbf{F}(\mathbf{u}) = \begin{bmatrix} 0 & -v \\ -v & 0 \end{bmatrix} \mathbf{u}$$

as an alternative way of expressing the (initial value PDE) flux-conservative equation

$$\partial \mathbf{u}/\partial t = -\partial \mathbf{F}(\mathbf{u})/\partial x$$ with **F** being the conserved flux.

You may recognize this description as an analog to Maxwell's equations for the 1D propagation of E&M waves.

<u>Advective Equation</u>

Suppose we have a problem similar in form to the *advective equation* (1D fluid flow at velocity v):

$$\partial u/\partial t = -v\partial u/\partial x$$

where the analytic solution is known (see text), but the numerical approach we'll learn is widely applicable to solving hyperbolic equations.

Approximate the time derivative of u by

$$\partial u/\partial t\,|_{j,n} = [u_j^{n+1} - u_j^n] / \Delta t + O(\Delta t)$$

$x_j = x_0 + j\Delta x, \qquad j=0,1,\ldots,J$

$t_n = t_0 + n\Delta t, \qquad n=0,1,\ldots,N$

We've seen this numerical approximation: it's called the ____.

Approximate the spatial derivative of u by

$$\partial u/\partial x\,|_{j,n} = [u_{j+1}^{n} - u_{j-1}^{n}] / 2\Delta x + O(\Delta x^2)$$

The advective equation can be numerically solved according to the Forward Time-Centered Space:

$$[u_j^{n+1} - u_j^{n}] / \Delta t \quad = \quad -v\ [u_{j+1}^{n} - u_{j-1}^{n}] / 2\Delta x$$

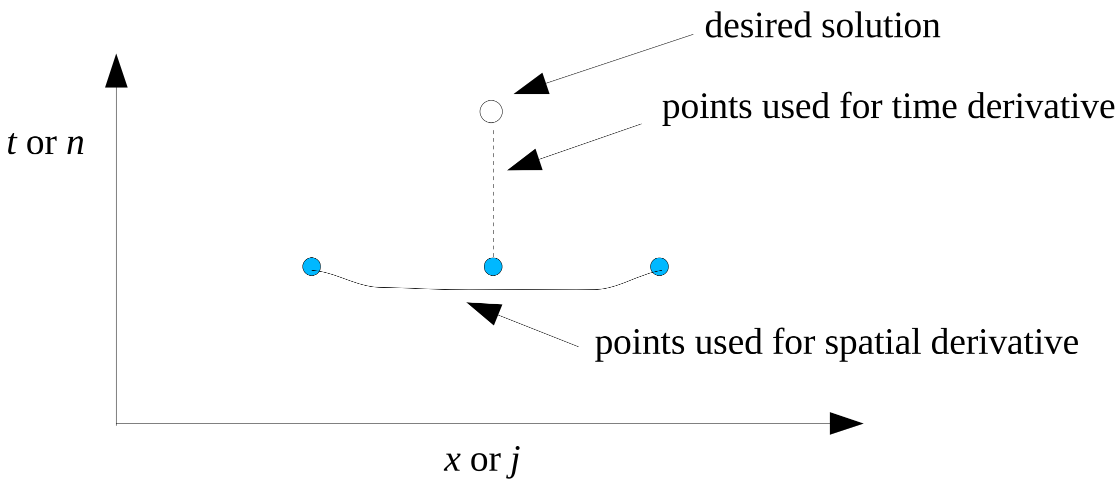

Diagram for FTCS

Question: for which variable are we trying to solve here? (indicated by the open circle) i.e., What are the subscript and superscript?

Unfortunately, as the book points out, the FTCS method does not do a good job! Mr. Lax found that replacing the u_j^n term with its spatial average fixes things, $u_j^n \rightarrow \frac{1}{2}(u_{j+1}^{n}+u_{j-1}^{n})$:

$$u_j^{n+1} = \tfrac{1}{2}(u_{j+1}^{n} + u_{j-1}^{n}) - v\,(u_{j+1}^{n} - u_{j-1}^{n})\,\Delta t/2\Delta x$$

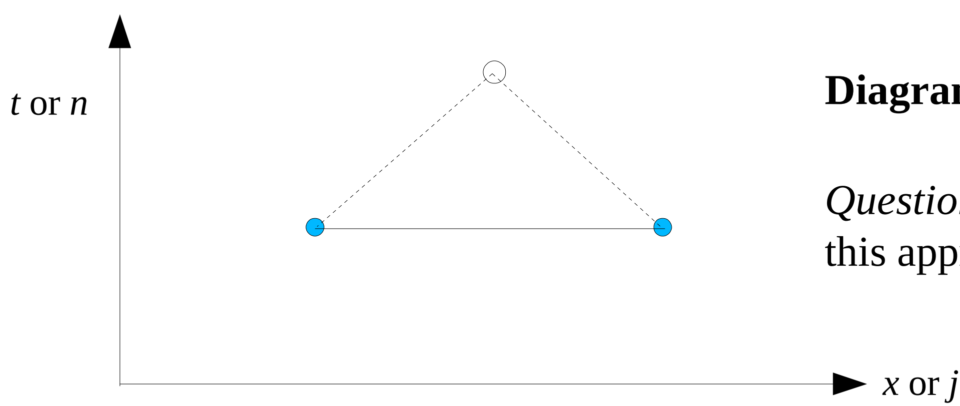

Diagram for Lax Differencing Approach

Question: What is the uncertainty level using this approach?