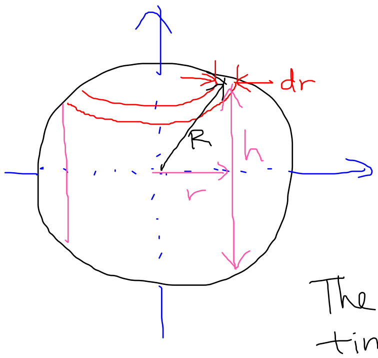
Moment of Inertia for a solid sphere of radius R + mass M



Divide the sphere into cylinders of radius r and thickness dr

The height of the cylinder varies with r:

$$\frac{h}{2} = \sqrt{R^2 - r^2} \Rightarrow h = 2\sqrt{R^2 - r^2}$$

The volume of the cylinder is its surface area times its thickness:

$$dV = (2\pi r)h\,dr = 4\pi r\sqrt{R^2 - r^2}\,dr$$

The sphere has uniform density $\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$,

so the mass of the cylinder is $dm = \rho\,dV = \frac{M}{\frac{4}{3}\pi R^3} 4\pi r\sqrt{R^2 - r^2}\,dr$

$$\Rightarrow dm = \frac{3M}{R^3} r\sqrt{R^2 - r^2}\,dr$$

The cylinder has moment of inertia $dI = r^2\,dm$

$$= \frac{3M}{R^3} r^3\sqrt{R^2 - r^2}\,dr$$

So, now we can find the total moment of inertia by summing up all the cylinders as r goes from 0 to R.

The integral is:

$$I = \int dI = \int_0^R \frac{3M}{R^3} r^3\sqrt{R^2 - r^2}\,dr = \frac{3M}{R^3}\int_0^R r^3\sqrt{R^2 - r^2}\,dr$$

To calculate this integral, I make the substitution $r = R\sin\theta$. when $r = 0$, $\theta = 0$ + when $r = R$, $\theta = \pi/2$. So my new limits of integration are 0 to $\pi/2$.

Also, $dr = d(R\sin\theta) = R\cos\theta\,d\theta$

$$I = \frac{3M}{R^3}\int_0^{\pi/2} R^3\sin^3\theta\sqrt{R^2 - R^2\sin^2\theta}\,R\cos\theta\,d\theta = \frac{3M}{\cancel{R^3}}\int_0^{\pi/2}\cancel{R^3}\sin^3\theta(R\cos\theta)R\cos\theta\,d\theta$$

$$= 3MR^2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta = 3MR^2\int_0^{\pi/2}\sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta$$

$$= 3MR^2\int_0^{\pi/2}(-\sin\theta\,d\theta)(-\cos^2\theta + \cos^4\theta)$$

$$= 3MR^2\left[-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right]\Big|_0^{\pi/2} = 3MR^2\left[0 - \left(-\frac{1}{3}+\frac{1}{5}\right)\right] = 3MR^2\left(\frac{2}{15}\right)$$

$$\boxed{I = \frac{2}{5}MR^2}$$