Damped oscillations

$$\sum F = \overbrace{-kx}^{\text{restoring force}} \overbrace{-\,bv}^{\text{friction/damping force}} = -kx - b\frac{dx}{dt}$$

$$\sum F = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

Solution has the form $x = Ce^{-at}$. What are a, C?

then $\frac{dx}{dt} = -aCe^{-at}$, $\frac{d^2x}{dt^2} = a^2Ce^{-at}$

Substituting: $m\left(a^2Ce^{-at}\right) = -kCe^{-at} + baCe^{-at}$

divide by Ce^{-at}: $ma^2 = -k + ba \Rightarrow ma^2 - ba + k = 0$

quadratic formula: $a = \frac{b \pm \sqrt{b^2 - 4km}}{2m} = \frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$

C is an arbitrary amplitude.

So, the full solution for $x(t)$ is

$$x = C_1e^{-a_1} + C_2e^{-a_2t} \quad \text{where} \quad a_1 = \frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

$$a_2 = \frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

If $\frac{b^2}{4m^2} > \frac{k}{m}$ $\left(b > 2\sqrt{km}\right)$, then a_1 + a_2 are real.

This is the OVERDAMPED case.

UNDERDAMPING:

If $\frac{b^2}{4m^2} < \frac{k}{m}$ $\left(b < 2\sqrt{km}\right)$, then a_1 & a_2 are COMPLEX.

Let's define $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ so that $a_1 = \frac{b}{2m} + i\omega'$

$a_2 = \frac{b}{2m} - i\omega'$

The general solution then is

$$x = C_1 e^{(-b/2m + i\omega')t} + C_2 e^{(-b/2m - i\omega')t}$$

where C_1 & C_2 are complex numbers. However, we are only interested in real solutions. So, we will make use of the fact that any complex number plus its conjugate is real. So, if $C_1 = a + ib$, then $C_2 = a - ib$ will give a real solution. We make use of the fact that any complex number may be written as

$$a + ib = Ae^{i\phi} = A(\cos\phi + i\sin\phi)$$

where $A = \sqrt{a^2+b^2}$, $\cos\phi = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\phi = \frac{b}{\sqrt{a^2+b^2}}$

Then, $C_1 = Ae^{i\phi}$ & $C_2 = Ae^{-i\phi}$

$$\Rightarrow x = Ae^{i\phi}e^{-bt/2m + i\omega' t} + Ae^{-i\phi}e^{-bt/2m - i\omega' t}$$

$$= Ae^{-b/2m}\left(e^{i\omega' t + i\phi} + e^{-i\omega' t - i\phi}\right)$$

$$= Ae^{-b/2m}\left\{\left[\cos(\omega' t + \phi) + i\sin(\omega' t + \phi)\right] + \cos(-\omega' t - \phi) + i\sin(-\omega' t - \phi)\right]\right\}$$

$$= Ae^{-b/2m}\left\{\left[\cos(\omega' t + \phi) + \cancel{i\sin(\omega' t + \phi)}\right] + \cos(\omega' t + \phi) - \cancel{i\sin(\omega' t + \phi)}\right]\right\}$$

$$x = 2Ae^{-b/2m}\cos(\omega' t + \phi)$$

Since A is an arbitrary amplitude, $\boxed{x = Ae^{-b/2m}\cos(\omega' t + \phi)}$ for the UNDERDAMPED case.