Astr 2310 Thurs. Feb. 16, 2016

Today's Topics

- Celestial Mechanics cont.
  - Newtonian Derivation of Kepler's Laws
  - Newton's Test of Universal Gravitation
  - The Two-Body Problem
  - Least Energy Orbits
  - Example of Least-Energy Orbit to Mars
- Chapter 2: Solar System Overview
  - Constituents
  - Discovery of Outer Planets
  - Fundamental Characteristics
    - Mass and Radius
    - Surface Temperature and Black Body Radiation
    - Planetary Atmospheres and Composition
  - Radioactivity and Half-Life
    - Nuclear Physics (see The Making of the Atomic Bomb by R. Rhodes)
    - Age Dating of Solar System

Homework this Week

- A2310 HW #2
- Due Thursday Feb. 18
- Ryden & Peterson: Ch. 2: #3, #4, #5
- Ryden & Peterson: Ch. 3: #1, #2, #4, #5, #6, #9

Geometric Properties of the Ellipse

$FF' = 2ae$ (definition of e)

Consider triangle BcF:

$b^2 + a^2e^2 = r^2 = a^2 \; (r + r' = 2a)$ so:

$b^2 = a^2 - a^2e^2 = a^2(1 - e^2)$

$b = a(1 - e^2)^{1/2}$ (relationship between b & a)

Furthermore:

$R_{min} = a - ae = a(1 - e)$

$R_{max} = a + ae = a(1 + e)$

(distances at perihelion, aphelion)

Applying law of cosines to F PF gives:

$r'^2 = r^2 + (2ae)^2 + 2r(2ae)\cos\theta$

But since $r' = 2a - r$ we have:

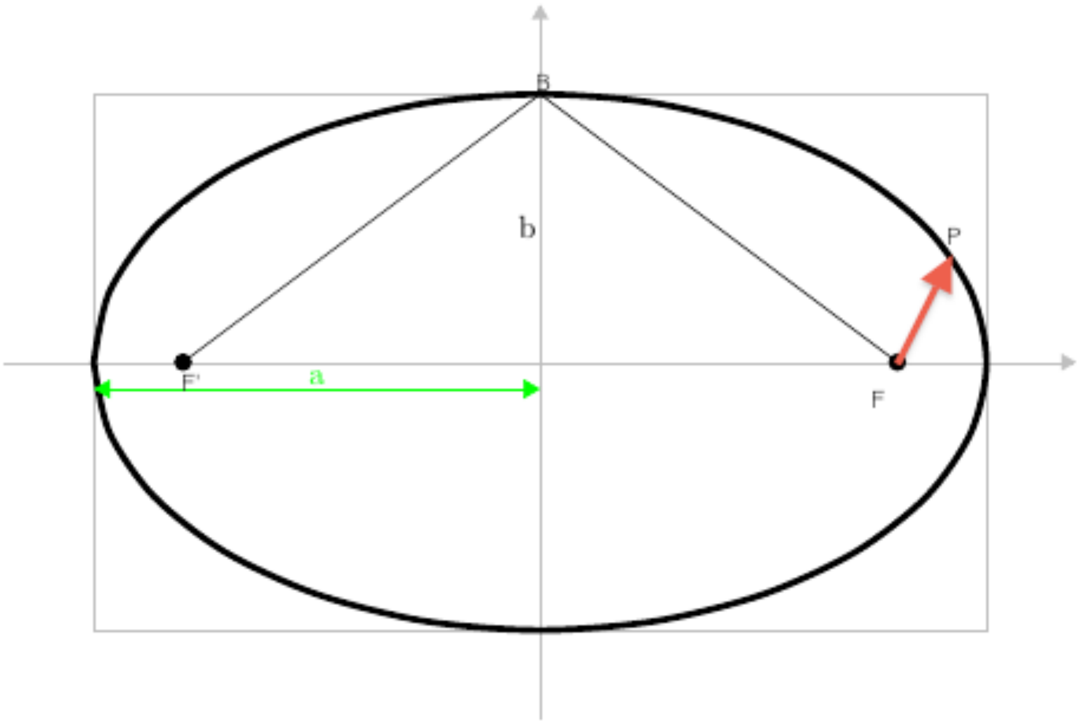
$4a^2 - 4ar + r^2 = r^2 + 4a^2e^2 + 4rae\cos\theta$

$a - r = ae^2 + re\cos\theta$

$a - ae^2 = r + re\cos\theta$

$a(1 - e^2) = r(1 + e\cos\theta)$ so:

$r = a(1 - e^2)/(1 + e\cos\theta)$ (equ. for ellipse in polar coordinates)



What About the Velocity?

Kepler's 2nd law:

$1/2\ r^2\ dq/dt\ =$ constant (must hold for entire period)

$1/2r^2\ dq/dt\ =\ \pi ab/P$ (area/period)

Since $b\ =\ a(1-e^2)^{1/2}$:

$r^2 dq/dt\ =\ (2\pi a/P)[a(1-e^2)^{1/2}]$

Or:

$d\theta/dt = (2\pi/P)(a/r)^2(1-e^2)^{1/2}$

Recall $s\ =\ rq$ so $ds/dt\ =\ r\ dq/dt\ =\ V\theta$

$V_\theta = r\ dq/dt\ =\ r(2\pi/P)(a^2/r^2)(1-e^2)^{1/2}$

$= (2\pi/P)[a^2(1-e^2)^{1/2}]/[a(1-e^2)/(1+e\cos\theta)]$

So finally:

$V_\theta = (2\pi a/P)(1+e\cos\theta)/(1-e^2)^{1/2}$

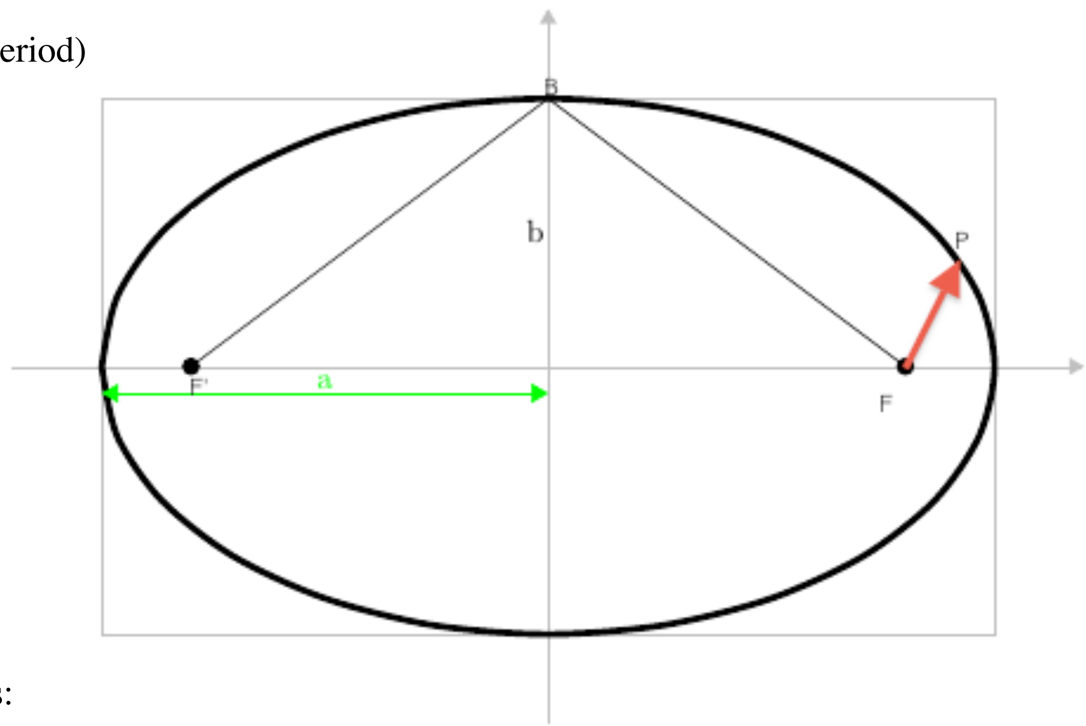
Since $1-e^2\ =\ (1+e)(1-e)$ so we consider 2 cases:

Perihelion velocity ($\theta\ =\ 0^o$):

$V_{peri} = (2\pi a/P)(1+e)/(1-e^2)^{1/2}$

Aphelion velocity ($\theta\ =\ 180^o$):

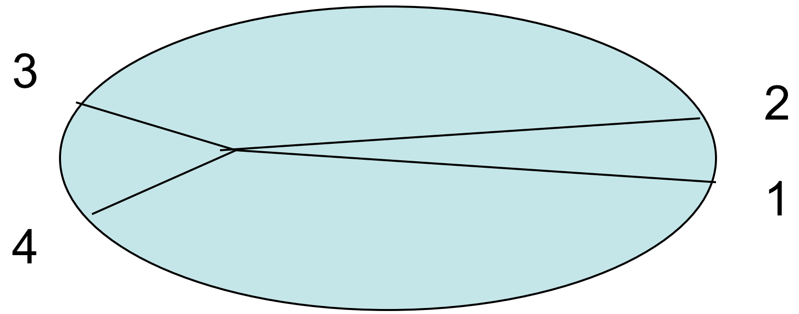
$V_{aph}\ =\ (2\pi a/P)(1-e)(1-e^2)^{1/2}$

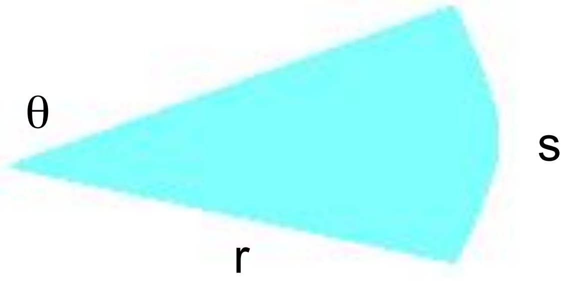


Newtonian Derivation of Kepler's Laws

- #1: The general form of a planetary orbit is an ellipse/conic section
  - Extensive derivation requiring calculus (see Mechanics)
- #2: A planet in orbit about the Sun sweeps out equal areas in equal amounts of time
  - Recall that the area of a sector is given by:

    Area = $\theta r^2/2$ (θ in radians)





Consider the motion of a planet between points 1 & 2 and between points 3 & 4. The orbital path length is given by s_1 and the angular difference is given by q and a given time interval:

$\Delta t \; = \; t_2 - t_1 \; = \; t_4 - t_3$:

The conservation of angular momentum requires:

$mv_1 r_1 \; = \; mv_2 r_2 \; = \; mv_3 r_3 \; = \; mv_4 r_4$ so:

$v_1 \mathrm{r}_1 = \mathrm{v}_3 \mathrm{r}_3$ and multiplying by Δt gives:

$\Delta \mathrm{tv}_1 \mathrm{r}_1 = \Delta \mathrm{tv}_3 \mathrm{r}_3$ but since distance = velocity x time we have:

$s_{12} r_1 = s_{34} r_3$ but $\mathrm{s}_{12} = r_1 \theta_{12}$ and $\mathrm{s}_{34} = r_3 \theta_{34}$ so:

$\theta_{12} \; r_{12} = \theta_{34} r_{32}$ dividing by 2 gives:

$(\theta_{12} r_{12}) / 2 = (\theta_{34} r_{32}) / 2$ (area of sectors)

2-nd law results from conservation of angular momentum.

Law #3: The square of the orbital period is proportional to the cube of the semi-major axis of it's orbit. Consider a circular orbit for simplicity. Equate the centripital and gravitational forces ($F_c = F_g$):

$M_p V_p^2) / r = (G M_s M_p) / r^2$ dividing by M_p and $1/r$:

$V_p^2 = G M_p / r$ but the circular velocity is:

$V_p = 2\pi r / P$ where P is the orbital period so:

$(2\pi)^2 r^2 / P^2 = G M_p / r$ and solving for p we have:

$P^2 = (4\pi^2 / G M_s) r^3$ but the circle is a special case of an ellipse so:

$P^2 = k\ a^3$ or $P^2 = a^3 / M$ (P is in years, a in AU and M is in solar masses)

Newton's Test of Universal Gravitation

Recall the form of Newton s Gravitational Law:

$F_g = GMm / r^2$ so $a_g = F_g / m = GM / r^2$

a (apple) = 9.807 m/s^2 (at R_E)

Since R_E = 6378 km and $d_m = 3.844 \times 10^5$ km:

R_E / d_m = 60.27 so the acceleration at dm should be:

$a_m = a_g /(60.27)^2 = a_g /3632$

But what is it?

$a_m = V_m^2 / d_m \quad V_m = (2\pi d_m)/P = 1.023 \times 10^3$ m/s

So $a_m = 2.723 \times 10\text{-}^3$ m/s^2

$a_g /3632 = 2.698 \times 10^{-3}$ m/s^2 (within 1%!)

Two-Body Problem

Center of Mass: location where Fg = 0, and lies along the line connecting the two masses. Each mass must have the same orbital period and so:

$P_1 = 2\pi r_1/v_1 = P_2 = 2\pi r_2/v_2$ so $r_1/v_1 = r_2/v_2$ and $r_1/r_2 = v_1/v_2$

Newton's 3rd law means $F_1 = F_2$ so:

$m_1 v_1^2/r_1 = m_2 v_2^2/r_2$ substituting for V gives:

$(m_1 4\pi^2 r_1^2)/r_1 P^2 = (m_2 4\pi^2 r_2^2)/r_2 P^2$ or:

$m_1 r_1 = m_2 r_2$ thus: $r_1/r_2 = m_2/m_1 = v_1/v_2$

Now we define a relative orbit where the more massive object, i.e., the Sun, lies near the center of mass. Let $a = r_1 + r_2$ and $v = v_1 + v_2$

Since $r_1 = r_2 + m_2/m_1$ and $r_1 + r_2 = m_2 r_2/m_1 + r_2$ so $a = r_2(1 + m_2/m_1)$

The displacement is small for planets. Note:

$r_2 = a/(m_1/m_1 + m_2/m_1)$ and $r_2 = m_1 a/(m_1 + m_2)$ combining gives:

$r_1 = m_2 a/(m_1 + m_2)$ (note the symmetry)

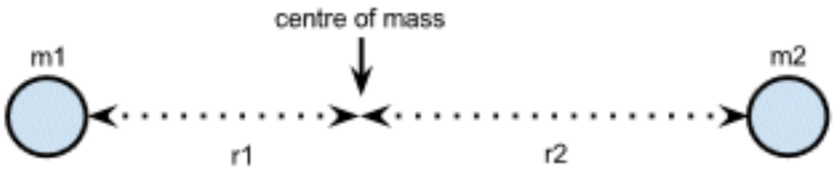
Recall that $F_g = F_c$ (gravity = centripital force)

$F_1 = m_1 v_1^2/r_1 = Gm_1 m_2/(r_1 + r_2)^2$ substituting for v_1 (circular orbit)

$4\pi^2 m_1 r_1/P^2 = 4\pi^2 m_1 m_2 a/P^2(m_1 + m_2) = Gm_1 m_2/a^2$ Thus:

$P^2 = [4\pi^2/G(m_1 + m_2)]a^3$ (Newtonian form of Kepler s 3-rd Law)

Note: Masses can be derived given the period and semi-major axis of the orbits.



Two-Body Problem

Center of Mass: location where Fg = 0, and lies along the line connecting the two masses. Each mass must have the same orbital period and so:

$P_1 = 2\pi r_1/v_1 = P_2 = 2\pi r_2/v_2$ so $r_1/v_1 = r_2/v_2$ and $r_1/r_2 = v_1/v_2$

Newton's 3rd law means $F_1 = F_2$ so:

$m_1 v_1^2/r_1 = m_2 v_2^2/r_2$ substituting for V gives:

$(m_1 4\pi^2 r_1^2)/r_1 P^2 = (m_2 4\pi^2 r_2^2)/r_2 P^2$ or:

$m_1 r_1 = m_2 r_2$ thus: $r_1/r_2 = m_2/m_1 = v_1/v_2$

Now we define a relative orbit where the more massive object, i.e., the Sun, lies near the center of mass. Let $a = r_1 + r_2$ and $v = v_1 + v_2$

Since $r_1 = r_2 + m_2/m_1$ and $r_1 + r_2 = m_2 r_2/m_1 + r_2$ so $a = r_2(1 + m_2/m_1)$

The displacement is small for planets. Note:

$r_2 = a/(m_1/m_1 + m_2/m_1)$ and $r_2 = m_1 a/(m_1 + m_2)$ combining gives:

$r_1 = m_2 a/(m_1 + m_2)$ (note the symmetry)

Recall that $F_g = F_c$ (gravity = centripital force)

$F_1 = m_1 v_1^2/r_1 = Gm_1 m_2/(r_1 + r_2)^2$ substituting for v_1 (circular orbit)

$4\pi^2 m_1 r_1/P^2 = 4\pi^2 m_1 m_2 a/P^2(m_1 + m_2) = Gm_1 m_2/a^2$ Thus:

$P^2 = [4\pi^2/G(m_1 + m_2)]a^3$ (Newtonian form of Kepler s 3-rd Law)

Note: Masses can be derived given the period and semi-major axis of the orbits.