P2310 Example Problems on Polarization Matrices

Recall that the polarization state of a beam of light can be specified via Jones matrices. In this case the “beam matrix” can be thought of as a 1 x 4 matrix and we can define polarization operators that are 4 x 4 matrices that represent various kinds of polarizers. Thus, when a beam matrix is multiplied by a polarization operator it represents a polarizer being placed in the beam and the new polarization state (beam matrix) can be calculated. For example:

Recall that polarization state of a wave can be specified by a 2 x 1 matrix:

\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\]

where A is the amplitude along the x-axis and B is the amplitude along the y-axis and \( A^2 + B^2 = 1 \)

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

linear polarization along x-dimension

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

linear polarization along y-dimension

\[
\begin{bmatrix}
\cos \alpha \\
\sin \alpha
\end{bmatrix}
\]

linear polarization at angle \( \alpha \) from x-axis

A 2 x 2 "operator" matrix can be used to represent a polarizer:

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

linear polarizer with transmission along x-axis

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

linear polarizer with transmission along y-axis

Example 1: Consider light polarized along the x-axis encountering:

a) linear polarizer oriented along the x-axis (horizontal). In this case:

\[
\begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

and so there is no effect, as expected.

b) linear polarizer oriented along the y-axis (horizontal). In this case:

\[
\begin{bmatrix}
A \\
B
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

and so no light passes through.

This looks kind of trivial and of little to no use unless one considers an arbitrarily polarized beam of light and a polarizer at an arbitrary angle \( \theta \) with respect to the x axis. If we define the polarizer’s transmission axis as 1 and its opaque axis as 2 then we use the expressions for a rotation of the coordinate system. In this case:
\[
\vec{x} = \cos \theta \vec{u}_1 - \sin \theta \vec{u}_2 \quad \text{and} \quad \vec{y} = \sin \theta \vec{u}_1 + \cos \theta \vec{u}_2 \quad \text{and} \quad \vec{E}_x = E_x \cos \theta + E_y \sin \theta \quad \text{and} \quad \vec{E}_y = -E_x \sin \theta + E_y \cos \theta \\
\vec{u}_1 = \cos \theta \vec{x} + \sin \theta \vec{y} \quad \text{and} \quad \vec{u}_2 = -\sin \theta \vec{x} + \cos \theta \vec{y}
\]

After some algebra we can show that:

\[
\begin{bmatrix}
  A \\
  B
\end{bmatrix} = \begin{bmatrix}
  \cos^2 \theta & \sin \theta \cos \theta \\
  \sin \theta \cos \theta & \sin^2 \theta
\end{bmatrix} \begin{bmatrix}
  E_x \\
  E_y
\end{bmatrix}
\]

For quarter-wave and half-wave plates the derivation is complicated and so we just give the results for being the angle of the fast axis relative to the x-axis:

\[
\begin{bmatrix}
  \cos^2 \theta + i \sin^2 \theta & \sin \theta \cos \theta - i \sin \theta \cos \theta \\
  \sin \theta \cos \theta - i \sin \theta \cos \theta & \sin^2 \theta + i \cos^2 \theta
\end{bmatrix} \quad \text{quarter-wave}
\]

\[
\begin{bmatrix}
  \cos^2 \theta - \sin^2 \theta & 2 \sin \theta \cos \theta \\
  2 \sin \theta \cos \theta & \sin^2 \theta - \cos^2 \theta
\end{bmatrix} \quad \text{half-wave}
\]

Consider two examples:

a) a quarter-wave plate with a fast axis at \( \theta = 45^\circ \) operating on horizontally polarized light:

\[
\frac{1}{\sqrt{2}} \begin{bmatrix}
  1 \\
  -i
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
  1 & -i \\
  -i & 1
\end{bmatrix} \begin{bmatrix}
  1 \\
  0
\end{bmatrix} \quad \text{(turns linear into right-circular)}
\]

b) a half-wave plate with a fast axis at \( \theta = 45^\circ \) operating on horizontally polarized light:

\[
\begin{bmatrix}
  0 \\
  1
\end{bmatrix} = \begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix} \begin{bmatrix}
  1 \\
  0
\end{bmatrix} \quad \text{(turns horizontally polarized to vertical)}
\]