

Phys 2310 Wed. Sept. 13, 2017

Today's Topics

- **Chapter 15: The Superposition of Waves**
 - **Interference**
 - **Methods for Adding Waves**
 - **Boundary Conditions**
 - **Standing Waves & Normal Modes**
- **Supplementary Material:**
 - **Fourier Analysis**
- **Reading for Next Time**

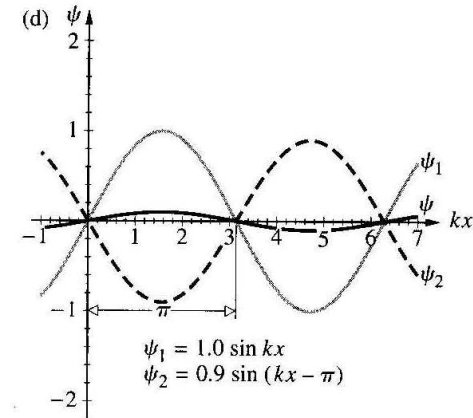
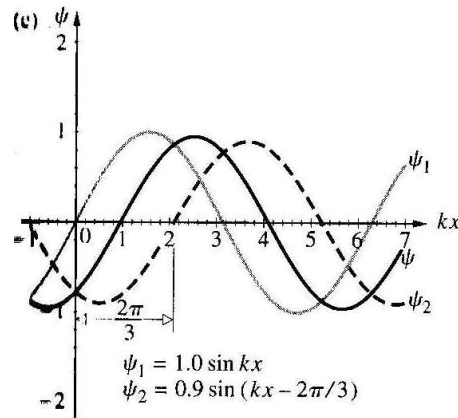
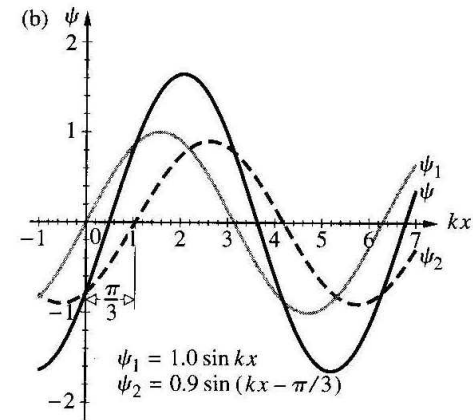
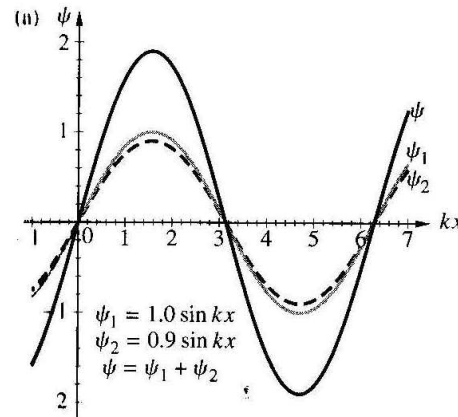
Homework this Week

**SZ Chapter 15: #22, 32, 35, 39, 40, 42, 46,
47**

Due Monday Sept. 18

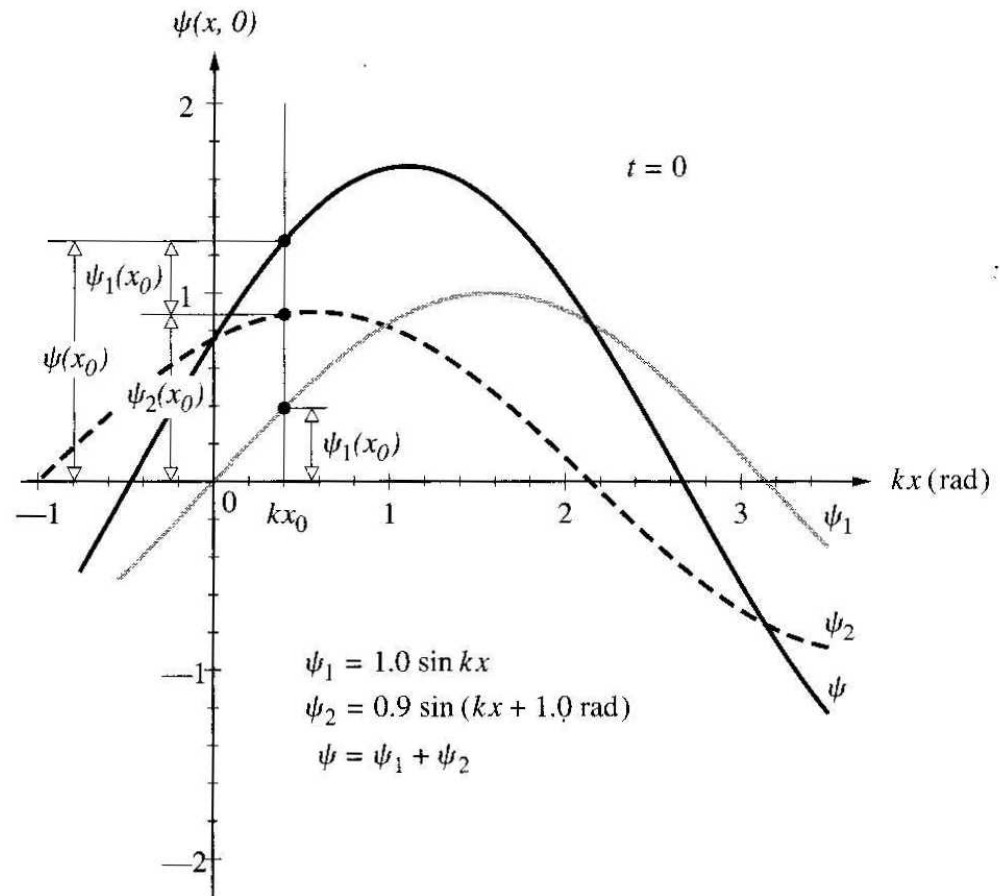
SZ Chapter15: Superposition Principle

- **Concept of Superposition**
 - The addition of two waves is also a wave that satisfies the wave equation
 - Result is the sum of the function (disturbance) at each point in space
- **Superposition Examples**
 - Consider two waves: $A_1 = 1.0$, $A_2 = 0.9$
 - Wave functions add according to their amplitudes and phases
 - Note: (a) – addition (constructive interference) (b) - almost cancellation (destructive interference)
- **Mathematical Techniques**
 - Write some software that evaluates waves at each point (x) and each time and adds them.



SZ Chapter 15: Superposition Principle

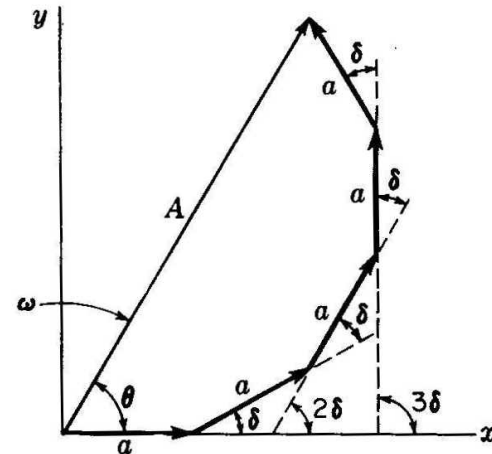
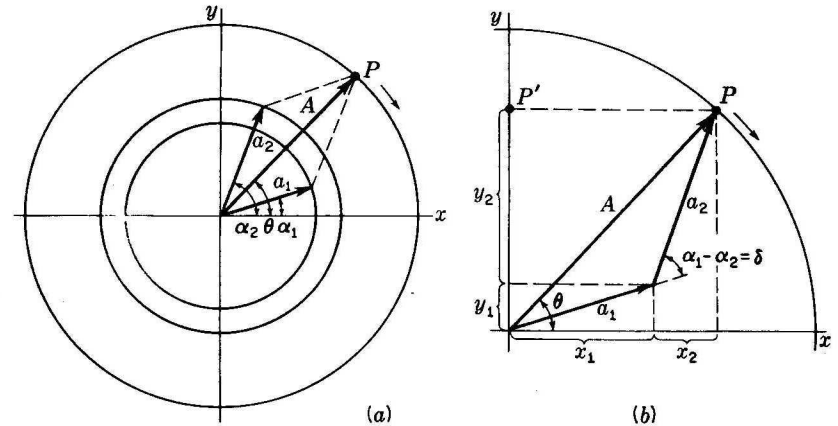
- **Superposition Examples cont.**
 - Consider two waves: $A_1 = 1.0$, $A_2 = 1.0$
 - Now waves have same amplitude and freq. but different phases
 - Wave functions add according to their amplitudes and phases as before.
 - That is, we could numerically compute the value of the wave at each value of kx and then add them together or we could make use of law of cosines since they have the same phase (next slide)
- **We Need a Better Technique for Adding Amplitudes and Phases of Waves**



SZ Chapter 15: Graphical Addition of Waves

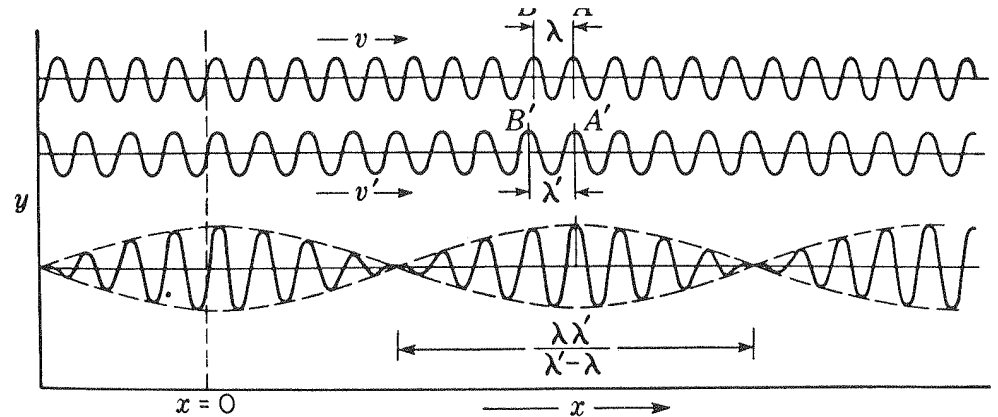
- **Law of Cosines:**
 - $A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\alpha_1 - \alpha_2)$
- **Polar coordinates can be used to graphically add amplitudes and phases.**
 - **Think in terms of vectors (phasors):**
 - $A < \phi$ or sometimes $A < \theta$
 - **Multiple waves:**
 - $a_1 < \delta_1, a_2 < \delta_2$, etc.
 - **Constructive interference:**
 - a_1 opposite a_2
 - **Constructive interference:**
 - a_1 parallel to a_2
 - **General interference:**
 - Compute components and add.
 - Compute new amplitude and phase.

See diagram at right.



Suppl: Group vs. Phase Velocity

- If two waves have different frequency the case is more complex
- For two harmonic waves with slightly different frequency:



$$\psi_1 = a \sin(\omega_1 t - k_1 x) \text{ and } \psi_2 = a \sin(\omega_2 t - k_2 x)$$

$$\psi = \psi_1 + \psi_2 = 2a \sin\left(\frac{\omega_1 + \omega_2}{2} t - \frac{k_1 + k_2}{2} x\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t - \frac{k_1 - k_2}{2} x\right)$$

The figure at right shows this result, a wave with 2x the amplitude (sin) and the average of the two wavelengths and frequencies but modulated by a second wave of low frequency and longer wavelength (cos). This produces "beating."

The phase velocity is thus :

$$v = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega_1}{k_1} \text{ as we might expect but the group velocity is :}$$

$$u = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega}{dk} \text{ (group velocity). Thus :}$$

$$u = v + k \frac{dv}{dk} = v - \lambda \frac{dv}{d\lambda}$$

Water waves show $\frac{dv}{d\lambda} \neq 0$ but for light in vacuum :

$$\frac{dv}{d\lambda} = 0 \text{ (light in vacuum)}$$

SZ Chapter 15: Addition with Same Frequency

- **Algebraic Method**

- **Recall that light waves are not affected when the beams cross.**
- **The disturbance (E-field amplitude) just adds locally.**

Consider two waves:

$$E_1 = E_{01} \sin(\omega t + \alpha_1) \quad \text{and} \quad E_2 = E_{02} \sin(\omega t + \alpha_2)$$

where $\alpha = -(kx + \varepsilon)$ is the spatial portion of the phase.

Adding the two waves (trig. identities):

$$E = E_{01} (\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + E_{02} (\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2)$$

and so when the spatial and time-dependent portions are separated:

$$E = (E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2) \sin \omega t + (E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2) \cos \omega t$$

Now note that the terms in () are constant in time. So we define:

$$E_0 \cos \alpha = E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2 \quad \text{and} \quad E_0 \sin \alpha = E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2$$

Squaring and adding to get the intensity:

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\alpha_2 - \alpha_1) \quad (\text{note interference term})$$

Dividing to get the phase:

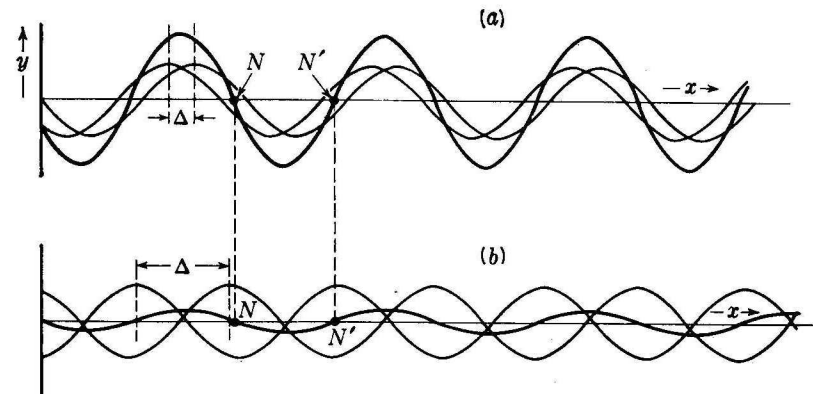
$$\tan \alpha = \frac{E_{01} \sin \alpha_1 + E_{02} \sin \alpha_2}{E_{01} \cos \alpha_1 + E_{02} \cos \alpha_2} \quad \text{thus:}$$

$$E = E_0 \cos \alpha \sin \omega t + E_0 \sin \alpha \cos \omega t$$

Note the interference term varies with OPL:

$$\delta = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2) \quad \text{or:}$$

$$\delta = \frac{2\pi}{\lambda_0} n(x_1 - x_2) \quad (\text{where } n(x_1 - x_2) \text{ is the OPL})$$



SZ Chapter 15: Standing Waves

- **Standing Waves: Special Case of Two Waves with the Same Frequency**

- **Boundary Conditions: Wave is Fixed at Both Ends with Phase Change Upon Reflection (see Figure 15.19)**

$y_1(x,t) = -A \cos(kx + \omega t)$ and $y_2(x,t) = A \cos(kx - \omega t)$ so adding:
 $y(x,t) = A[-\cos(kx + \omega t) + \cos(kx - \omega t)]$ and using the trig. identity:
 $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ we get:
 $y(x,t) = (2A \sin kx) \sin \omega t$ (note the modulated amplitude)

Note that since both have the same frequency the result will also have the same frequency, only the amplitude changes with x position.

Nodes occur where the amplitude goes to 0. Requiring:

$L = n \frac{\lambda}{2}$ for a string fixed at both ends (n is any integer). Solving for λ :

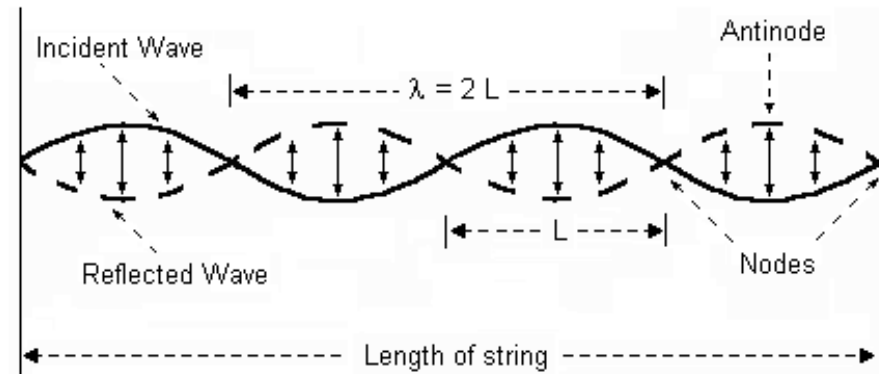
$\lambda_n = \frac{2L}{n}$ The corresponding frequencies ($f = v / \lambda$) are:

$f_n = \frac{v}{2L}$ with the wave velocity depending on mass/length and tension.

Thus the string can also vibrate in harmonics:

$$y_n(x,t) = A \sin k_n x \sin \omega t$$

String instruments are designed to vibrate in the fundamental mode and the first few harmonics since pure tones sound like old-school video game music.



Electron wave resonance

$n = 1$
 $\lambda_1 = 2\pi r_1 = 6.28a_0$

$n = 2$
 $2\lambda_2 = 2\pi r_2$
 $\lambda_2 = 12.57a_0$

$n = 3$
 $3\lambda_3 = 2\pi r_3$
 $\lambda_3 = 18.85a_0$

String resonance modes

$n = 1$
 Fundamental
 $\lambda_1 = 2L$

$n = 2$
 $\lambda_2 = L$

$n = 3$
 $\lambda_3 = \frac{2L}{3}$

SZ Chapter 15: Addition with Same Frequency

• Standing Waves

- Oscillating string constrained at both ends has $A = 0$ at endpoints.
- Weiner showed that the E field at the surface of a mirror must be zero, just like the E&M theory of a conductor.
- Reflected waves are like standing waves with a phase change of π .

In order to satisfy this boundary condition consider both waves:

$$E = E_I + E_R = E_{0I}[\sin(kx + \omega t) + \sin(kx - \omega t)]$$

And since:

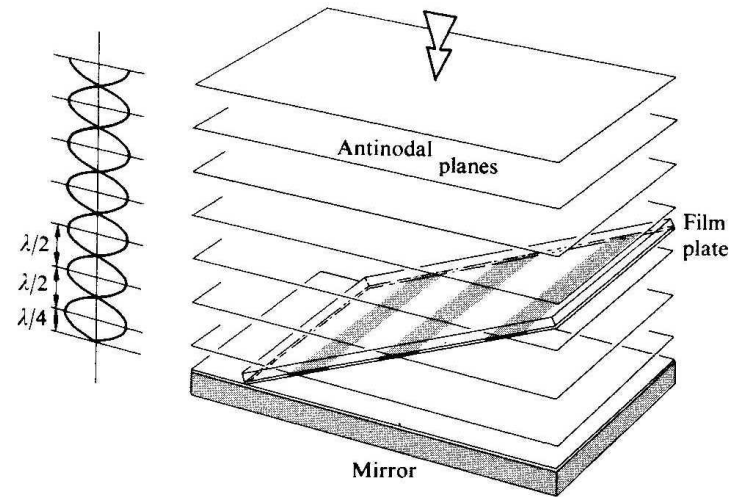
$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta) \text{ we have:}$$

$$E(x,t) = 2E_{0I}\sin kx\cos\omega t \text{ (standing wave)}$$

Note that one part oscillates in time but the amplitude or "envelope" doesn't, hence the name. This results in nodes and anti-nodes.

In addition, Weiner's experiment showed that reflected light can interfere.

This technique can be used to measure the wavelength of light from the OPD.



Supplementary: 2-d Harmonic Oscillator

• Now Consider a 2-d Harmonic Oscillator

$$x = A_1 \cos(\omega_1 t + \alpha_1) \text{ and } y = A_2 \cos(\omega_2 t + \alpha_2)$$

Since the $\cos()$ functions vary between +1 and -1 the x displacement is confined within $\pm A_1$ and the y displacement is confined within $\pm A_2$. Thus any point: $P(x, y, t)$ will be confined within the corresponding rectangle.

If $\omega_1 \neq \omega_2$ or $\frac{\omega_1}{\omega_2} \neq$ ratio of whole numbers then the pattern

will never repeat and eventually fill in the entire region. For the special case where $\omega_1 = \omega_2$:

$$x = A_1 \cos(\omega t) \text{ and } y = A_2 \cos(\omega t + \delta) \text{ consider a few cases:}$$

1 : $\delta = 0$. In this case: $x = A_1 \cos(\omega t)$ and $y = A_2 \cos(\omega t)$ and so:

$$y = \frac{A_2}{A_1} x \text{ and so P traces out a diagonal line.}$$

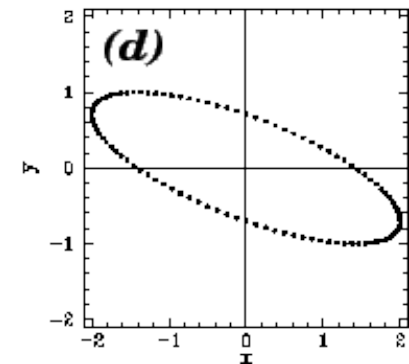
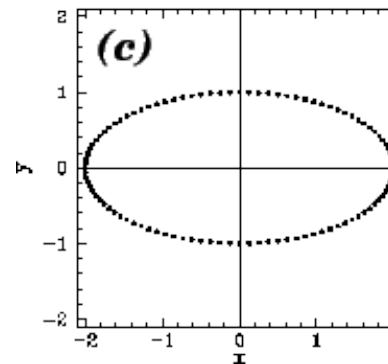
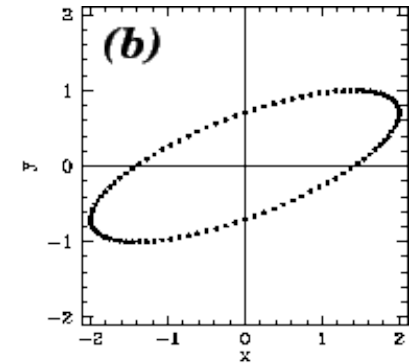
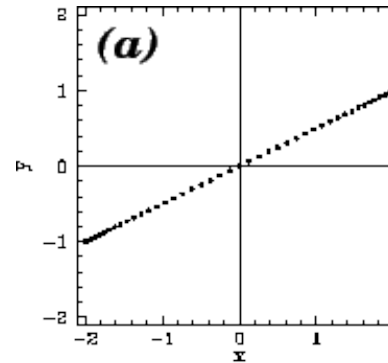
2 : $\delta = \pi / 2$ and so now we have:

$$x = A_1 \cos(\omega t) \text{ and } y = A_2 \cos(\omega t + \delta) = -A_2 \sin(\omega t) \text{ and since:}$$

$\sin^2 \omega t + \cos^2 \omega t = 1$ we must have:

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \text{ which is the equation of an ellipse with axes along}$$

the x and y axes. Note further that as t begins to increase x decreases while y goes negative. Thus the vector between the origin and P rotates clockwise.

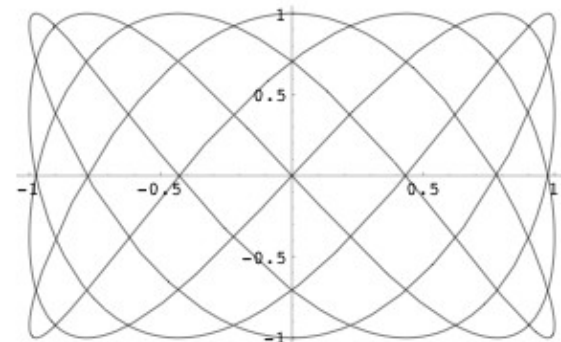
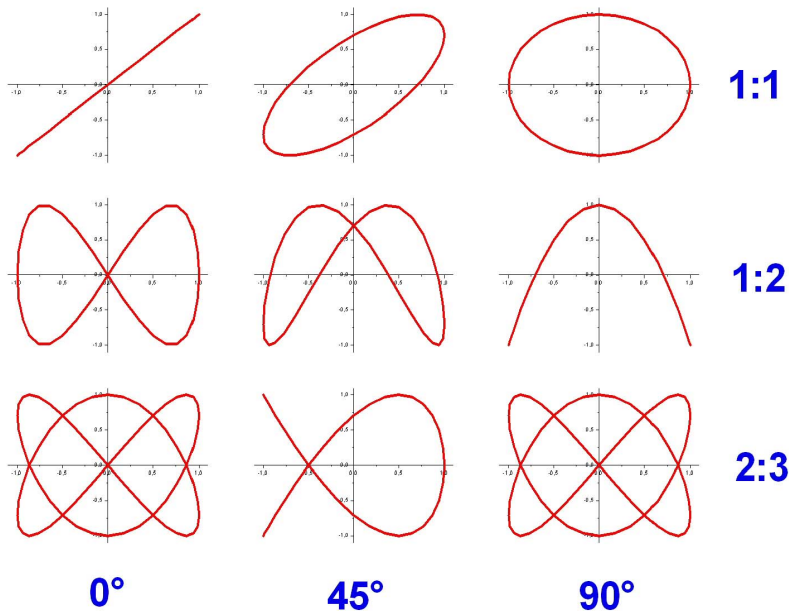


Supplementary: 2-d Harmonic Oscillator

• 2-d Harmonic Oscillator Continued

3: If $\frac{\omega_1}{\omega_2}$ is the ratio of whole numbers we see complex but repeating pattern.

Consider the case where $\omega_2 = 2\omega_1$. During one cycle of ω_2 we go through only one half cycle of ω_1 . The result is the family of curves shown below depending on the phase offset between the two wavefunctions.



SZ Chapter 15: Addition of Many Waves with Same Frequency

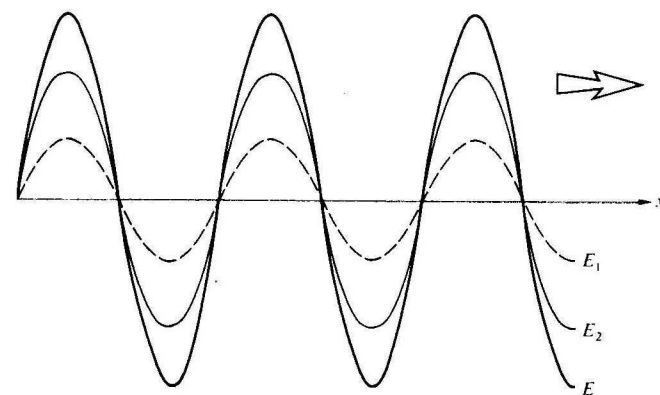
- **Superposition of Many Waves**
 - Now consider that case of many waves:

$$E = \sum_{i=1}^N E_{0i} \cos(\alpha_i \pm \omega t) = E_0 \cos(\alpha \pm \omega t) \text{ where:}$$

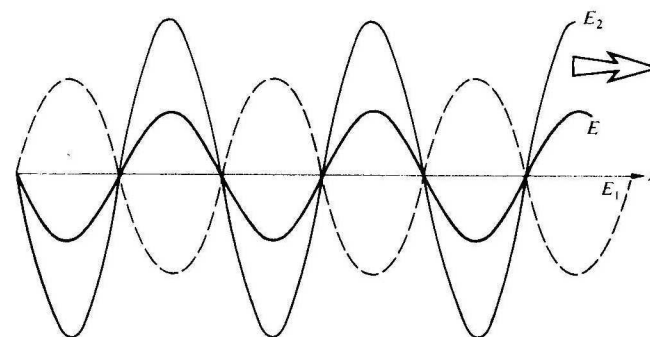
$$E_0^2 = \sum_{i=1}^N E_{0i}^2 + 2 \sum_{j>1}^N \sum_{i=1}^N E_{0i} E_{0j} \cos(\alpha_1 - \alpha_2) \text{ and:}$$

$$\tan \alpha = \frac{\sum_{i=1}^N E_{0i} \sin \alpha_i}{\sum_{i=1}^N E_{0i} \cos \alpha_i}$$

Note that since all have the same frequency the result will also have the same frequency, only the amplitude changes.



$$E = E_1 + E_2$$



Suppl.: Addition of Many Waves with Same Frequency

• Complex Method and Phasor Addition

- Recall that Phasor addition is like vector addition in polar coords.

If $E_1 = E_{01}e^{i(\alpha_1 \mp \omega t)}$ and there are N waves :

$$E = \left[\sum_{j=1}^N E_{0j} e^{i\alpha_j} \right] e^{i\omega t} \quad \text{where}$$

$$E_0 e^{i\alpha} = \left[\sum_{j=1}^N E_{0j} e^{i\alpha_j} \right] \quad \text{is known as the complex amplitude.}$$

Recall that since $E_0^2 = (E_0 e^{i\alpha})(E_0 e^{i\alpha})^*$ and so for N = 2 :

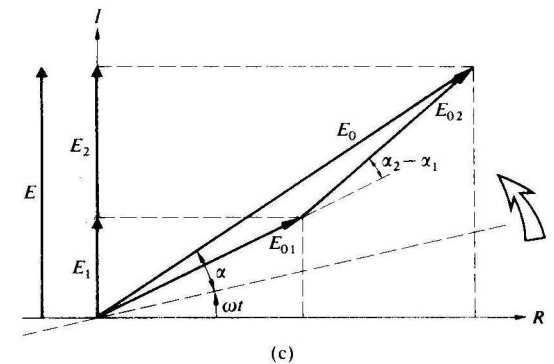
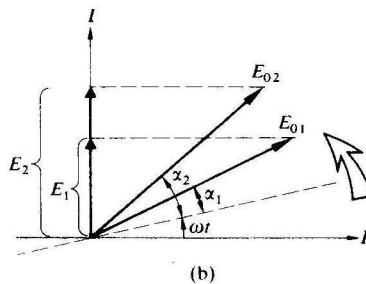
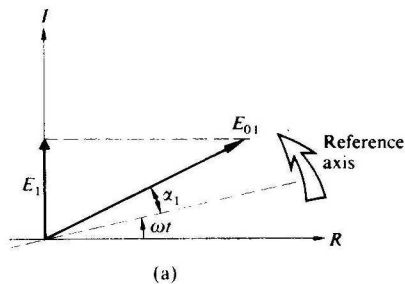
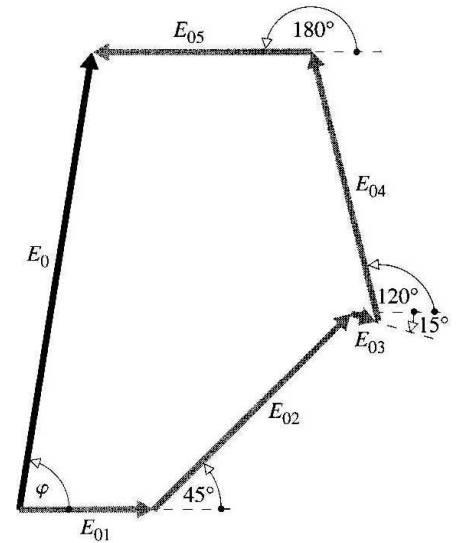
$$E_0^2 = (E_{01} e^{i\alpha_1} + E_{02} e^{i\alpha_2})(E_{01} e^{-i\alpha_1} + E_{02} e^{-i\alpha_2})$$

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}[e^{i(\alpha_1 - \alpha_2)} + e^{-i(\alpha_1 - \alpha_2)}]$$

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\alpha_1 - \alpha_2)$$

Similarly for Phasor addition the law of cosines gives :

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\alpha_1 - \alpha_2)$$



Suppl.: Addition of Many Waves with Same Frequency

- **Complex Method and Phasor Addition**
 - Consider the Phasor addition of 5 waves

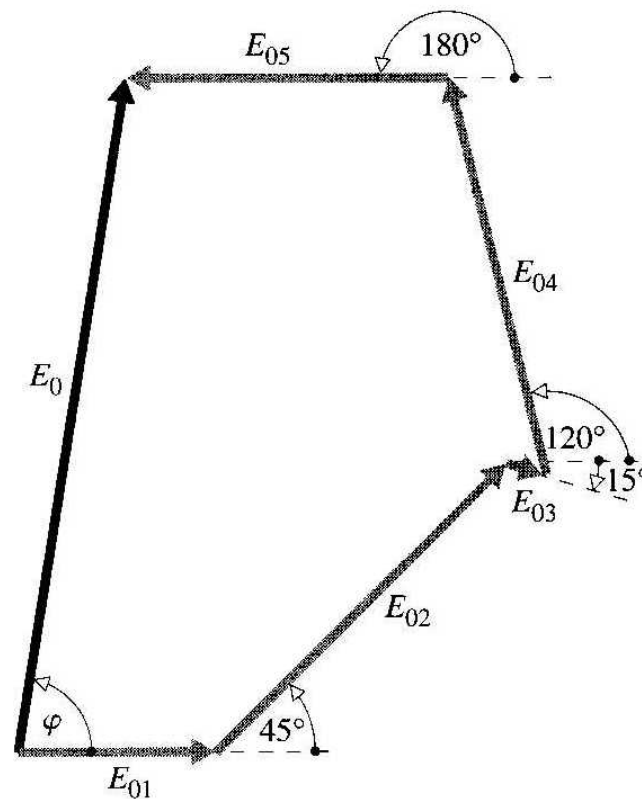
$$E_1 = 5 \sin \omega t$$

$$E_2 = 10 \sin(\omega t + 45^\circ)$$

$$E_3 = \sin(\omega t - 15^\circ)$$

$$E_4 = 10 \sin(\omega t + 120^\circ)$$

$$E_5 = 8 \sin(\omega t + 180^\circ)$$



SZ Chapter 15: Addition with Different Frequency

• Beats from Two Waves of Different Frequency

Consider the addition of two waves of different frequency :

$$E_1 = E_{01} \cos(k_1 x - \omega_1 t) \text{ and } E_2 = E_{02} \cos(k_2 x - \omega_2 t)$$

If the amplitudes are the same their addition yields :

$$E = E_{01} [\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)] \text{ and since :}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \text{ we have :}$$

$$E = 2E_{01} \cos \frac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t] \times \cos \frac{1}{2}[(k_1 - k_2)x - (\omega_1 - \omega_2)t]$$

Note that this wave has a frequency equal to the average of the two but is modulated by a much lower frequency wave $(\omega_1 - \omega_2)$:

If we re-cast the differences in frequency and propagation number :

$$\omega_m \equiv \frac{1}{2}(\omega_1 - \omega_2) \text{ and } \bar{\omega} \equiv \frac{1}{2}(\omega_1 + \omega_2) \text{ and :}$$

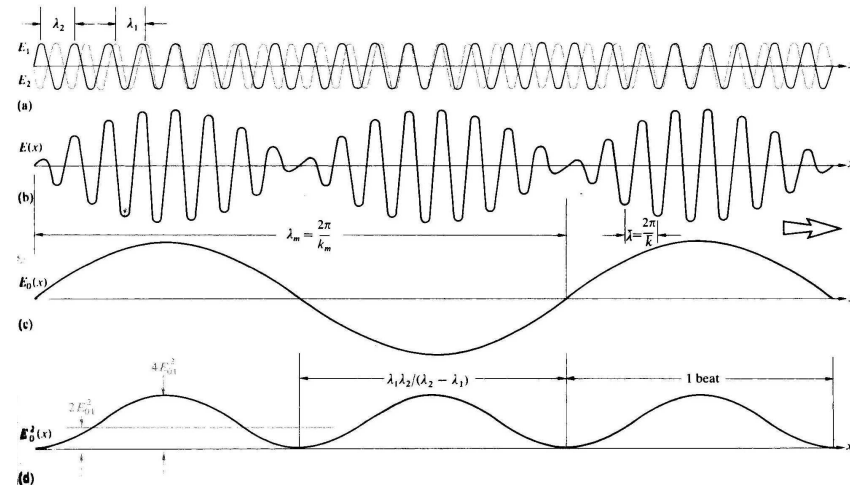
$$k_m \equiv \frac{1}{2}(k_1 - k_2) \text{ and } \bar{k} \equiv \frac{1}{2}(k_1 + k_2) \text{ we have :}$$

$$E = 2E_{01} \cos(k_m x - \omega_m t) \cos(\bar{k} x - \bar{\omega} t)$$

Note that the irradiance becomes :

$$E_0^2 = 4E_{01}^2 \cos^2(k_m x - \omega_m t) \text{ which is :}$$

$$E_0^2 = 2E_{01}^2 [1 + \cos(2k_m x - 2\omega_m t)]$$



French: Anharmonic Periodic Waves

- **Fourier Series**

- **Recall that we stated that any periodic function can be synthesized by as sum of harmonic waves (sines and cosines). This is Fourier's Theorm. Sines and cosines are used because they are orthogonal, i.e., independent (see text on linear algebra).**

$$f(x) = C_0 + C_1 \cos\left(\frac{2\pi}{\lambda}x + \varepsilon_1\right) + C_2 \cos\left(\frac{2\pi}{\lambda/2}x + \varepsilon_2\right) + \dots$$

In the figure at right see how 6 harmonic functions can generate the complicated result, and vice versa.

Traditionally we use both cosines and sines since sines are odd and can provide a phase shift too :

If $k = 2\pi / \lambda$ then we can reformulate as :

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx \quad (\text{sum over wavevectors})$$

where the amplitudes can be computed by integrating both sides and noting orthogonality :

$$\int_0^{\lambda} \cos akx \cos bkx dx = \frac{\lambda}{2} \delta_{ab} \quad \text{and} \quad \int_0^{\lambda} \sin akx \sin bkx dx = \frac{\lambda}{2} \delta_{ab}$$

where δ_{ab} is 0 if $a \neq b$ and 1 if $a = b$ (Kronecker delta). Thus we find :

$$A_m = \frac{2}{\lambda} \int_0^{\lambda} f(x) \cos mkx dx \quad \text{and} \quad B_m = \frac{2}{\lambda} \int_0^{\lambda} f(x) \sin mkx dx$$

So, given a function we can compute the Fourier amplitudes to model it. Note the symmetry between the function and the amplitudes of the Fourier series. In addition a pure even function : $f(x) = f(-x)$ will contain only cosines and a purely odd function : $f(-x) = -f(x)$ will only have sine terms.

Note that a complex waveform can then be "modeled" just from the Fourier coefficients (A_m, B_m).

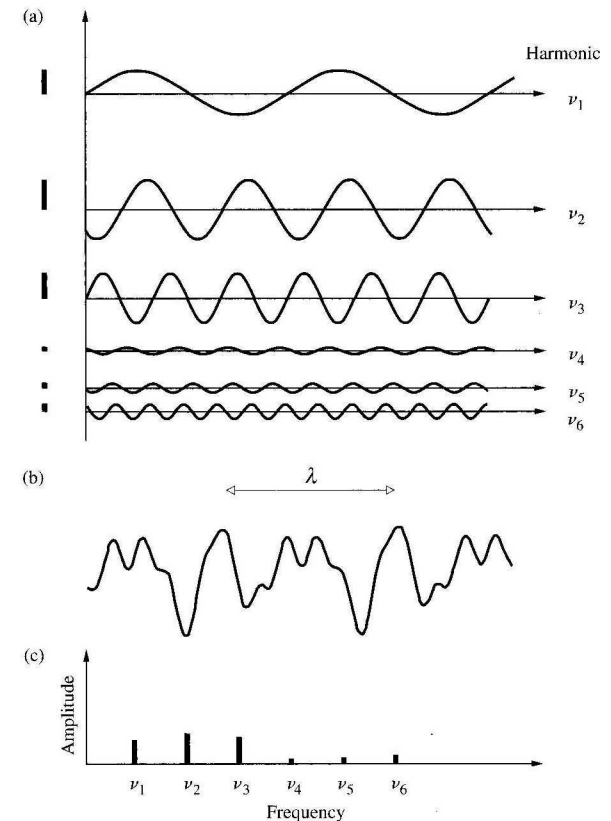
This greatly reduces a digital file size and is the basis of all compression algorithms. All trade

"fidelity" vs. file size. A familiar example is MP3. When we model the waveform, i.e., take a

Fourier transform, many times per second (576 for early versions of MP3), we end up with 576

sets (spectra) of Fourier amplitudes each second. The frequency range we sample (spectrum)

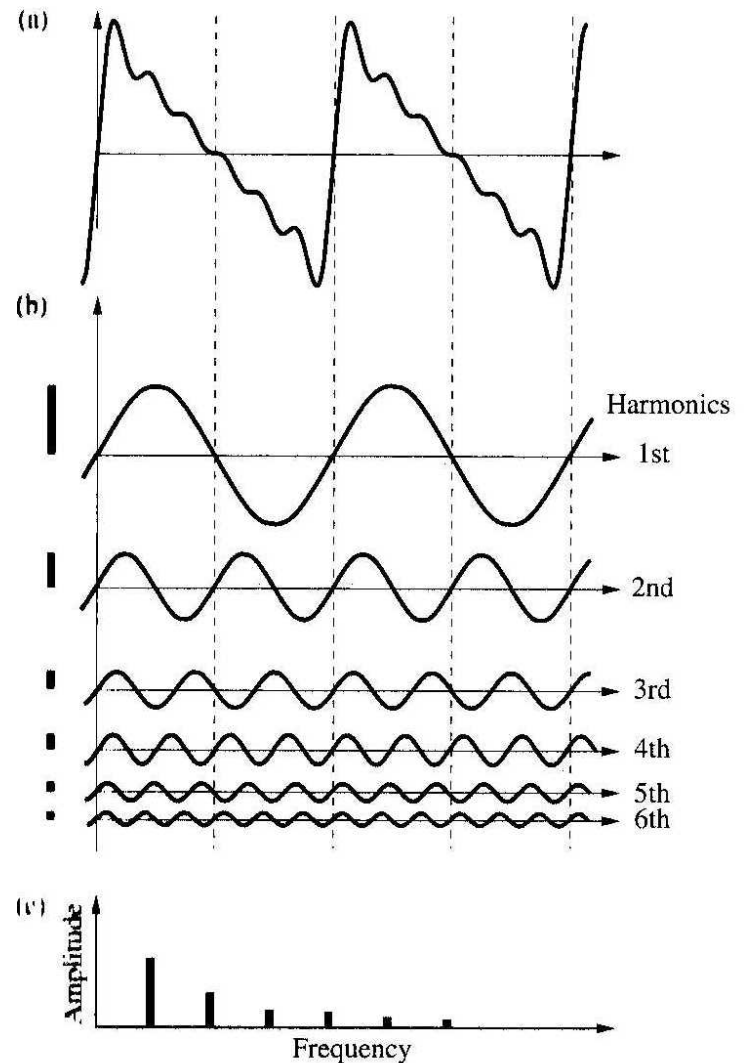
depends on the application. Lots of software is available for recording and converting audio to MP3.



French: Anharmonic Periodic Waves

- **Fourier Series**

- **Knowing a function, $f(x)$, we can compute the coefficients. The number of terms depends on the precision required.**
- **Note how only 6 terms (in sine) can reproduce the sawtooth function rather well since it is odd.**
- **Adding more terms is easy with a computer (even Excel).**
- **A few hundred terms may be necessary to accurately reproduce both the slopes and the discontinuities. The number required depends on the precision required.**
- **Note the decreasing amplitude of the higher-order terms in the frequency spectrum (typical).**



French: Anharmonic Periodic Waves

- **Fourier Series Example**

- Lets follow along with the book with an example square-wave.

$$f(x) = \begin{cases} +1 & \text{when } 0 < x < \lambda/2 \\ -1 & \text{when } \lambda/2 < x < \lambda \end{cases} \text{ so using only sines } (A_m = 0, \text{ odd function}) :$$

$$B_m = \frac{2}{\lambda} \int_0^{\lambda/2} +1 \sin mkx dx + \frac{2}{\lambda} \int_{\lambda/2}^{\lambda} -1 \sin mkx dx \text{ and thus :}$$

$$B_m = \frac{1}{m\pi} [-\cos mkx]_0^{\lambda/2} + \frac{1}{m\pi} [\cos mkx]_{\lambda/2}^0 \text{ or :}$$

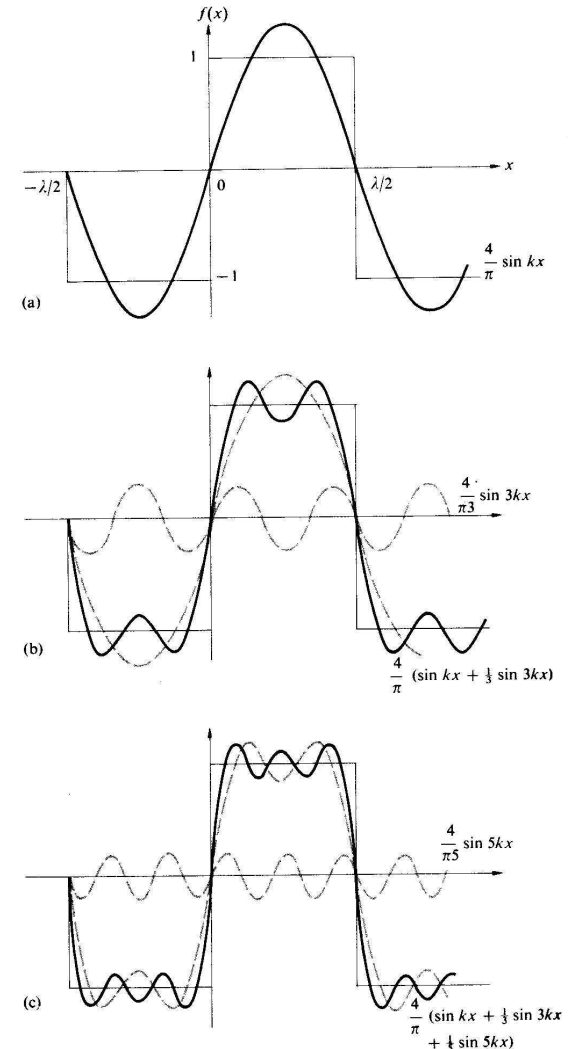
$$B_m = \frac{2}{m\pi} (1 - \cos m\pi) \text{ since } k = 2\pi/\lambda. \text{ Thus the Fourier coefficients are :}$$

$$B_1 = 4/\pi, B_2 = 0, B_3 = 4/3\pi, B_4 = 0, B_5 = 4/5\pi \text{ and so on. Thus :}$$

$$f(x) = \frac{4}{\pi} \left(\sin kx + \frac{1}{3} \sin 3kx + \frac{1}{5} \sin 5kx + \dots \right)$$

Note the decreasing amplitude of the higher - order terms.

See Figure 7.21 and text for a description of how as the peak width narrows higher order harmonics are necessary to model the wave. That is, reproducing small features requires more Fourier coefficients (amplitudes).



French: Non-periodic Waves

- **Fourier Integrals**

- **If we accept that we can model any function with an infinite series of sines and cosines it seems reasonable to generalize Fourier series to Fourier Integrals. See figure 7.21 for how In this case we have the Fourier Transform:**

$$f(x) = \frac{1}{\pi} \left[\int_0^{\infty} A(k) \cos kx dk + \int_0^{\infty} B(k) \sin kx dk \right] \text{ with :}$$

$$A(k) = \int_{-\infty}^{\infty} f(x) \cos kx dx \text{ and } B(k) = \int_{-\infty}^{\infty} f(x) \sin kx dx$$

An interesting Fourier transform pair is the square wave

$$f(x) = E_0 \text{ when } |x| < L/2 \text{ otherwise } 0.$$

Its transform is the sinc function :

$$A(k) = E_0 L \frac{\sin(kL/2)}{kL/2}$$

Supplement: Pulses and Wave Packets

- **Square Pulse**

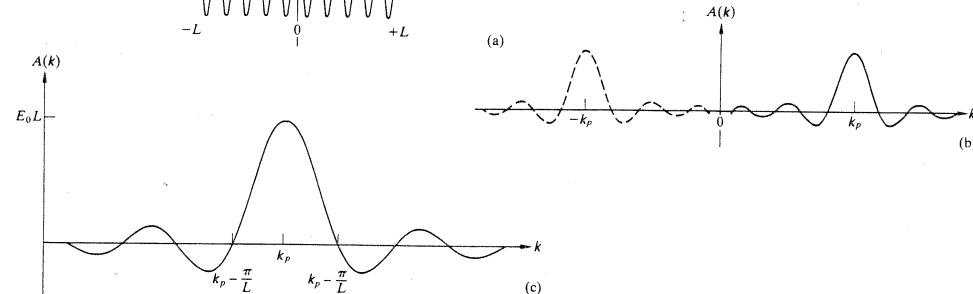
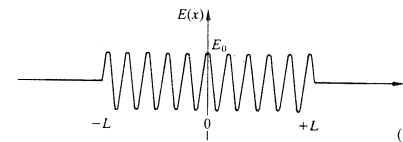
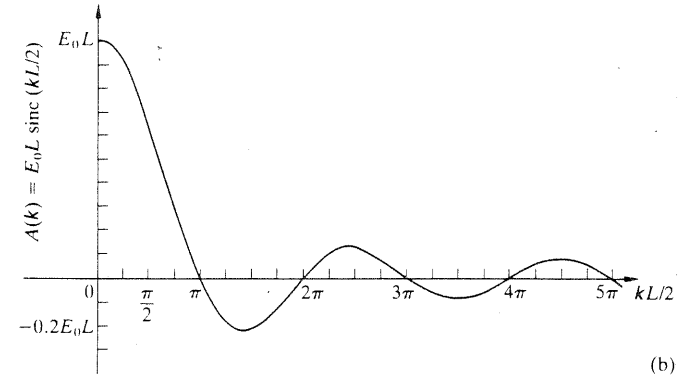
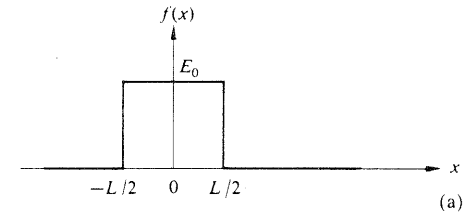
- Fourier transform of a square pulse is called the Sinc function

- **Finite Wavetrain**

- A given Fourier component goes on forever
- A finite wavetrain requires additional Fourier modes.
 - Think of the wave being modulated by a square pulse.
 - Thus a finite wavetrain (pulse) can be thought of as a wave packet.
 - As the wavetrain becomes long its frequency spectrum shrinks and vice versa.

- **Coherence Length**

- Any real E-M wave is not absolutely monochromatic but has a natural width.
- There must be a coherence length and a coherence time as a result.
 - Namely: $\Delta l_c = c\Delta t_c$
- Coherence length and time are a measure of space and time over which the wave has an approximately constant wavelength or frequency.
- Note that white light (large bandwidth) can then be understood as the superposition of large numbers of monochromatic waves (many Fourier modes)



Supplement: Non-periodic Waves

- **Discrete Fourier Transform**
 - Often our signal consists of a set of discrete points (e.g. from a digital detector)
 - The Fourier integral is approximated by a summation
 - The resulting transform can then be filtered to remove unwanted frequencies (e.g. interference) and then transformed back.
 - This techniques can be further generalized to muti-dimensional data, such as a 2-d images
 - See spatial filtering examples on the web.
 - **Two excellent references:**
 - **The Fourier Transform and Its Application (Bracewell)**
 - **A Student's Guide to Fourier Transforms: with Applications in Physics and Engineering (James)**

Summary and Key Concepts

- **Waves Add Together via Superposition**
- **Phasors Provide a Means for Adding Amplitudes and Phases**
- **Simple Cases for Just Two Waves**
- **Fourier Analysis for Adding Many Waves**
 - **We Can Model Any Wave as a Series of Sines and Cosines (Basis Set)**
- **Finite Wave Packets Require Frequency Range**

Reading this Week

By Friday:

Finish Ch. 15: Addition of Waves of the Same Frequency, Addition of Waves of Different Frequency

Supplement: Anharmonic Periodic Waves and Fourier Techniques

Homework this Week

**SZ Chapter 15: #22, 32, 35, 39, 40, 42, 46,
47**

Due Monday Sept. 18