

**Phys 2310 Mon. Oct. 19, 2009**

**Today's Topics**

- **Begin Chapter 2: Wave Motion**
- **Reading for Next Time**
- **No Lab this week!**

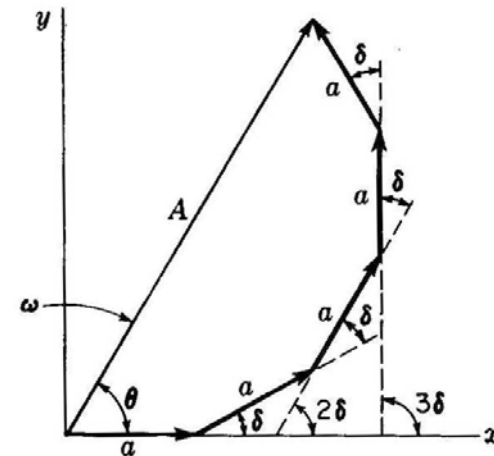
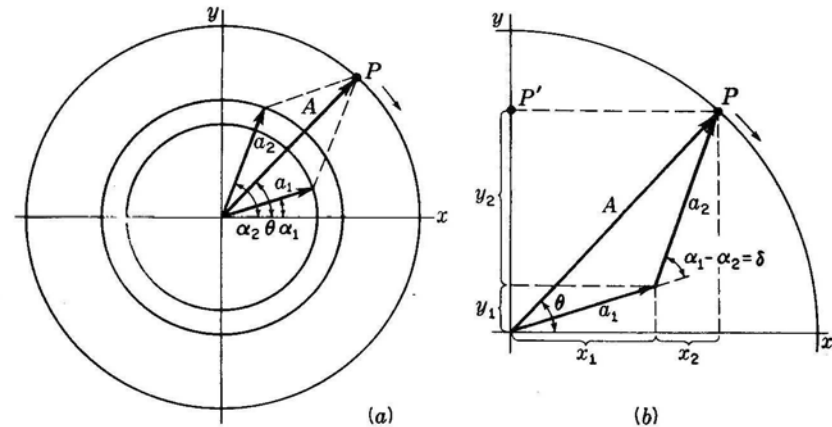
## **Homework this Week**

**Chapter 2: #3, 9 (use Excel), 17, 37, 38, 41**

**Due Fri. Oct. 23**

# Chapter 2: Graphical Addition of Waves

- **Polar coordinates can be used to graphically add amplitudes and phases.**
  - Think in terms of vectors (phasors):
    - $A < \phi$  or sometimes  $A < \theta$
    - Multiple waves:
      - $a_1 < \delta_1, a_2 < \delta_2$ , etc.
  - **Constructive interference:**
    - $a_1$  opposite  $a_2$
  - **Constructive interference:**
    - $a_1$  parallel to  $a_2$
  - **General interference:**
    - See diagram at right.



# Chapter 2: Complex Representation

- **Harmonic waves can also be described by complex numbers.**
  - **Main motivation is that the math is easier.**

$$\tilde{z} = x + iy \text{ where } i = \sqrt{-1}$$

$x$  and  $y$  are the real and imaginary parts of  $\tilde{z}$

Recall that  $x$  and  $y$  are the projections in polar coords.:

$$x = r \cos \theta \text{ and } y = r \sin \theta \text{ and so:}$$

$$\tilde{z} = r(\cos \theta + i \sin \theta) \text{ but since for } r = 1:$$

$$d\tilde{z} = i\tilde{z}d\theta \text{ we can integrate this and see:}$$

$$\tilde{z} = e^{i\theta} \text{ and so:}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \text{ and } e^{-i\theta} = \cos \theta - i \sin \theta \text{ so:}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \text{ and:}$$

$$\tilde{z} = re^{i\theta} \text{ (} r \text{ is amplitude and } \theta \text{ is the phase)}$$

# Chapter 2: Complex Representation

- **Harmonic waves can also be described by complex numbers.**
  - **Main motivation is that the math is easier.**
  - **Another justification of the Euler equation can be seen via a Taylor expansion of Sine and Cosine:**

Recall the Taylor expansion of the Sine :

$$\sin(\theta) \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

and for Cosine :

$$\cos(x) \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

Since the general form of an imaginary number is :

$$\tilde{z} = r[\cos(\theta) + i \sin(\theta)] \quad \text{then to a fourth - order :}$$

$$\tilde{z} \approx r \left[ 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \right) \right]$$

Expanding and grouping terms :

$$\tilde{z} \approx r \left[ 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} \right]$$

Note that the odd terms contain  $i$  but so do the even ones since  $i^2 = -1$ . Thus :

$$\tilde{z} \approx r \left[ 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} \right]$$

Now recall that the Taylor expansion for  $e^x$  is :

$$e^x = \left[ 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \right] \quad \text{and so :}$$

$$\tilde{z} = re^{i\theta} \quad (\text{indeed the Euler equation})$$

# Chapter 2: Complex Representation cont.

- **Mathematics of complex numbers**
  - **Absolute value, addition, multiplication, division**

Consider the complex conjugate :

Here we replace  $i$  with  $-i$ . Thus :

$$\tilde{z}^* = (x + iy)^* = (x - iy) \text{ and so :}$$

$$\tilde{z}^* = r(\cos \theta - i \sin \theta) \text{ and :}$$

$$\tilde{z}^* = r e^{-i\theta}$$

Addition of complex numbers :

$$\tilde{z}_1 \pm \tilde{z}_2 = (x_1 + iy_1) \pm (x_2 + iy_2) \text{ and so :}$$

$$\tilde{z}_1 \pm \tilde{z}_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

Multiplication and division becomes :

$$\tilde{z}_1 \tilde{z}_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \text{ and } \tilde{z}_1 / \tilde{z}_2 = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Absolute value (modulus) is :

$$\left| \tilde{z} \right| \equiv (\tilde{z} \tilde{z}^*)^{1/2} \text{ and so :}$$

$$\left| e^{\tilde{z}} \right| = e^x$$

# Chapter 2: Complex Representation cont.

- **Mathematics of complex numbers**
  - **Absolute value, addition, multiplication, division**

It is also clear that :

$$e^{i2\pi} = 1 \text{ since } \cos(2\pi) = 1 \text{ and } \sin(2\pi) = 0 \text{ and :}$$

$$e^{i\pi} = e^{-i\pi} = -1 \text{ and } e^{\pm i\pi/2} = \pm i$$

The real and imaginary parts of a complex number :

$$\operatorname{Re}\left(\tilde{z}\right) = r \cos(\theta) \text{ and } \operatorname{Im}\left(\tilde{z}\right) = r \sin(\theta) \text{ and so :}$$

$$\operatorname{Re}\left(\tilde{z}\right) = \frac{1}{2}\left(\tilde{z} + \tilde{z}^*\right) \text{ and } \operatorname{Im}\left(\tilde{z}\right) = \frac{1}{2i}\left(\tilde{z} - \tilde{z}^*\right)$$

Note that both the real and imaginary parts are harmonic

Traditionally the real part is used. Thus :

$$\psi(x, t) = \operatorname{Re}\left[Ae^{i(\omega t - kx + \varepsilon)}\right] = Ae^{i\varphi}$$

# Chapter 2: Three Dimensional Waves

- **Flat, plane waves**
  - **Wavefront: locus of points where disturbance has constant phase**
  - **Simplest form for a 3-d wave is a plane wave**
    - **Cartesian coordinates is sufficient**

We seek the equation for a plane perpendicular to a vector  $\vec{k}$ .

The position vector for any two points in the plane is :

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad (x, y, z) \quad \text{and} \quad \vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k} \quad (x_0, y_0, z_0)$$

and so a vector in the plane is :

$$(\vec{r} - \vec{r}_0) = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$$

Since  $\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$  the dot product of  $(\vec{r} - \vec{r}_0)$  and  $\vec{k}$  is zero :

and defines the plane is perpendicular to  $\vec{k}$  :

$$(\vec{r} - \vec{r}_0) \cdot \vec{k} = 0 \quad \text{or :}$$

$$k_x(x - x_0) + k_y(y - y_0) + k_z(z - z_0) = 0$$

But this requires :

$$k_x x + k_y y + k_z z = \text{const.} = k_x x_0 + k_y y_0 + k_z z_0$$

and so more generally the plane perpendicular to  $\vec{k}$  is :

$$\vec{k} \cdot \vec{r} = \text{const.} \quad \text{and so harmonic plane waves are :}$$

$$\psi(\vec{r}) = A \sin(\vec{k} \cdot \vec{r}) \quad \text{or} \quad \psi(\vec{r}) = A \cos(\vec{k} \cdot \vec{r}) \quad \text{or} \quad \psi(\vec{r}) = A e^{i(\vec{k} \cdot \vec{r})}$$

But for an infinite wave filling all space the wave repeats. Thus :

$$\psi(\vec{r}) = \psi\left(\vec{r} + \frac{\lambda \vec{k}}{k}\right) \quad \text{where } \vec{k} \text{ is the propagation vector and } k \text{ is the propagation number.}$$

## Chapter 2: Three Dimensional Waves cont.

- **Flat, plane waves**
  - Now consider the wave after some time  $t$
  - The phase moves with the phase velocity
  - In exponential form we have:

$$\psi(\vec{r}, t) = Ae^{i(\vec{k} \cdot \vec{r} \mp \omega t)}$$

Recall that since  $e^{i\lambda k} = 1 = e^{i2\pi}$  :

$k = 2\pi / \lambda$  Thus the wave velocity :

$$\frac{dr_k}{dt} = \pm \frac{\omega}{k} = \pm v$$

A less compact form but one emphasizing Cartesian coords is :

$$\psi(x, y, z, t) = Ae^{i(k_x x + k_y y + k_z z \mp \omega t)} \quad \text{which is equivalent to :}$$

$$\psi(x, y, z, t) = Ae^{i[k(\alpha x + \beta y + \gamma z) \mp \omega t]} \quad \text{where } \alpha, \beta, \gamma \text{ are the direction cosines.}$$

## Chapter 2: Three Dimensional Waves cont.

- **The Three-dimensional Wave Equation**

- **From the one-dimensional wave equation it is easy to see that:**

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

A more compact (and general) form uses the Laplacian operator :

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \text{ and so the wave equation becomes :}$$

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \text{ Note that this form does is not explicitly Cartesian.}$$

It can be shown that in spherical coordinates  $(r, \theta, \phi)$  :

$$\nabla^2 \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

but we can choose our origin to be the origin of the wave and so :

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) \text{ and so the equation becomes :}$$

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

Similarly, for cylindrical coordinates the Laplacian is :

$$\nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and so for cylindrically symmetric waves the wave equation becomes :

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$$

# Reading this Week

**By Wednesday:**

**Finish Ch. 2 (2.4 – 2.10) Superposition, Phasor addition, Complex representation, Multi-dimensional waves**

## **Homework this Week**

**Chapter 2: ##3, 9 (use Excel), 17, 37, 38, 41**

**Due Fri. Oct. 23**