Phys 2310 Fri. Nov. 17, 2017

Today' s Topics

- Begin Chapter 5: Diffraction
- Reading for Next Time

Reading this Week

By Mon.:

Read French Ch 8 from pages 288 – 302.

General Considerations,

Fraunhofer Diffraction, Fresnel Diffraction

Homework this Week

No HW this week. Study for the final.

Chapter 15: Diffraction

- General Considerations
  - There is no real physical distinction between diffraction and interference
  - Huygens-Fresnel Principle
    - Modification to Huygens Principle:

    Every unobstructed point of a wave-front, at any given instant, serves as a source for spherical, secondary wavelets (with the same frequency as that of the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering both their amplitudes and phases). This is really a quantum mechanical effect since there is otherwise no physical explanation.
  - Opaque Obstructions
    - When an aperture is large compared to λ, the effects of the boundary is minimal since the fraction of waves affected is small and vice versa.
    - Formally, diffraction occurs as a result of the boundary conditions for Maxwell's equations. The math is difficult but is solvable for a few special cases.
    - Aperture: imagine a set of fictitious, non-interacting oscillators distributed over the opening. Electrons at the edge interact with the E-field and dampen the EM wave.

  - Fraunhofer and Fresnel Diffraction
    - Fresnel diffraction: when the distance between aperture and screen/detector is small. Shape of wavefront is important.
    - Fraunhofer diffraction: when the distance between aperture and screen/detector is large ($R > a^2/\lambda$). This allows the assumption of plane waves.

Chapter 5: Diffraction

- **General Considerations**
  - **Light from Several Coherent Oscillators (artificial but instructive)**
    - **E-field will add according to amplitude and phase:**

$\tilde{E} = E_0(r)e^{i(kr_1-\omega t)} + E_0(r)e^{i(kr_2-\omega t)} + E_0(r)e^{i(kr_3-\omega t)} + \ldots\ + E_0(r)e^{i(kr_N-\omega t)}$ or:

$\tilde{E} = E_0(r)e^{ikr_1}e^{-i\omega t} \times [1 + e^{ik(r_2-r_1)} + e^{ik(r_3-r_1)} + e^{ik(r_4-r_1)} + \ldots e^{ik(r_N-r_1)}]$

but the phase difference arising from adjacent oscillators is simply:

$\delta = k(r_2 - r_1),\ 2\delta = k(r_3 - r_1),$ etc., where $\delta = kd\sin\theta$.

Thus the resulting field is:

$\tilde{E} = E_0(r)e^{ikr_1}e^{-i\omega t} \times [1 + (e^{i\delta}) + (e^{i\delta})^2 + (e^{i\delta})^3 + (e^{i\delta})^4 + \ldots (e^{i\delta})^{N-1}]$

The quantity in [] is a gemoetric series:

$[\,] = (e^{i\delta N} - 1)/(e^{i\delta} - 1) = e^{i(N-1)\delta/2}\left(\dfrac{\sin N\delta/2}{\sin\delta/2}\right)$ thus if r is the distance from center to P:

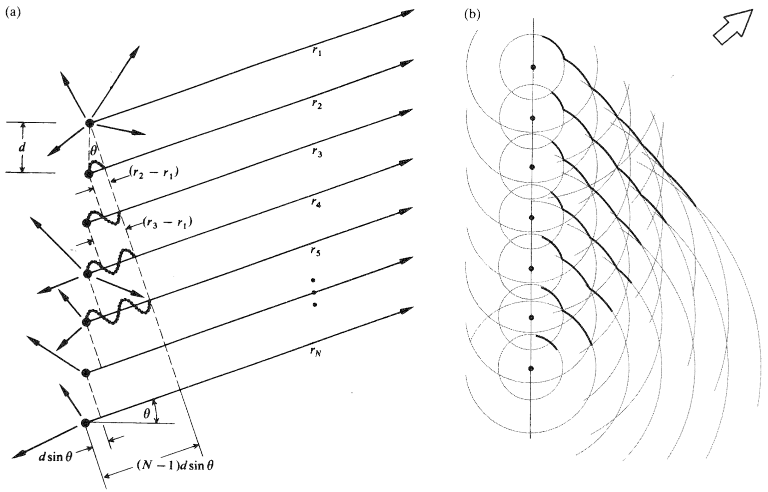
$\tilde{E} = E_0(r)e^{-i\omega t}e^{i[kr_1+(N-1)\delta/2]}\left(\dfrac{\sin N\delta/2}{\sin\delta/2}\right)$ and so the intensity is thus:

$I = I_0\dfrac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}$ but for $\delta = 2m\pi$ we have maxima at:

$d\sin\theta_m = m\lambda$ since $\left[\dfrac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}\right] = N^2$ for $\delta = 2m\pi$.

Note that for the special case of two slits we have:

$I = 4I_0\cos^2(\delta/2)$ as we saw before. Note too that electrons within optical materials surround atoms so close to each other that $d << \lambda$ so we experience only one maxima.



Chapter 5: Diffraction

- **Complications**
  - **Small angle: r ~ R – ysinθ ~ R - y**
  - **Example of a single slit**
    - **Position of maxima depend on λ**

Consider a slit element (ds) at the origin. Each element of the slit contributes to the field on the screen at perpendicular distance R. But the contribution from each point depends on the phase from that element:

$\delta = kd\sin\theta + \varepsilon$, where ε can be an intrinsic phase offset between adjacent slit elements. This is the concept of a "phased array" of radars. The maxima now occur at:

$d\sin\theta = m\lambda - \varepsilon / k$. This of the central maximum ($m = 0$) as the maxima of the ouput beam. Its orientation can be varied by adjusting ε and the radar can then "scan." Reversing the beam we see how an array of radio telescopes can simulate a "filled aperture" since each telescope (slit element) detects a phase offset. It the "elements" are distributed in two dimensions a large aperture can be simulated by combining the beams.

Returning to the screen, each slit element produces a field given by:

$E = \left(\frac{E_0}{r}\right)\sin\left(\omega t - kr\right)$ where r is the actual distance of P from each element and so:

$E = E_0 \int_{-D/2}^{D/2} \frac{\sin[\omega t - kr]}{r} dy$ where $r = r(y)$:

$r = R - y\sin\theta + (y^2/2R)\cos^2\theta + \ldots$

$E = \frac{E_L}{R}\frac{\sin[(kD/2)\sin\theta}{(kd/2)\sin\theta}\sin(\omega t - kR)$ but if $\beta = (kD/2)\sin\theta$ and $k = 2\pi/\lambda$ then:

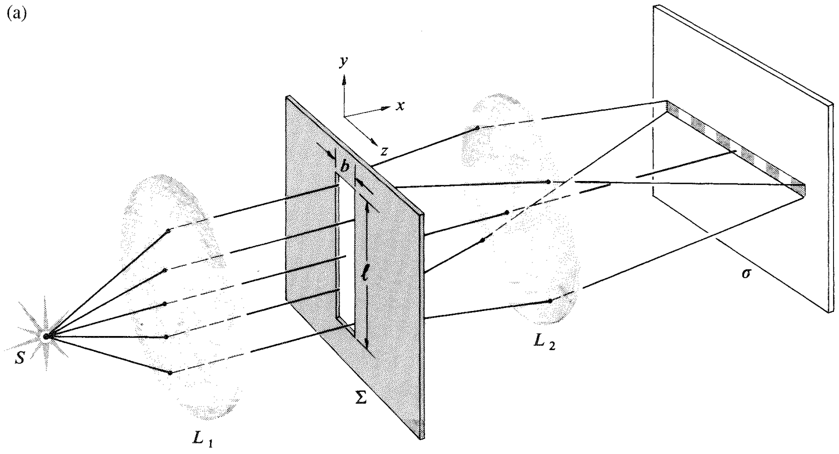
$E = \frac{E_L D}{R}\left(\frac{\sin\beta}{\beta}\right)\sin(\omega t - kR)$ but intensity is the square of the amplitude:

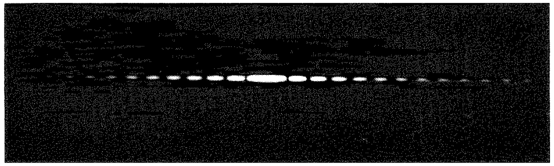
$I(\theta) = \frac{1}{2}\left(\frac{E_L D}{R}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2$ since $\left\langle \sin^2(\omega t - kR)\right\rangle = 1/2$. Simplifying:

$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$ which is the sinc function:

$I(\theta) = I(0)\sin c^2\beta$ and it can be differentiated to find the maxima and minima:

$\frac{dI}{d\beta} = I(0)\frac{2\sin\beta(\beta\cos\beta - \sin\beta)}{\beta^3} = 0$ or when $\sin\beta = 0$. Thus $\beta = \pm\pi, \pm 2\pi, \pm 3\pi\ldots$

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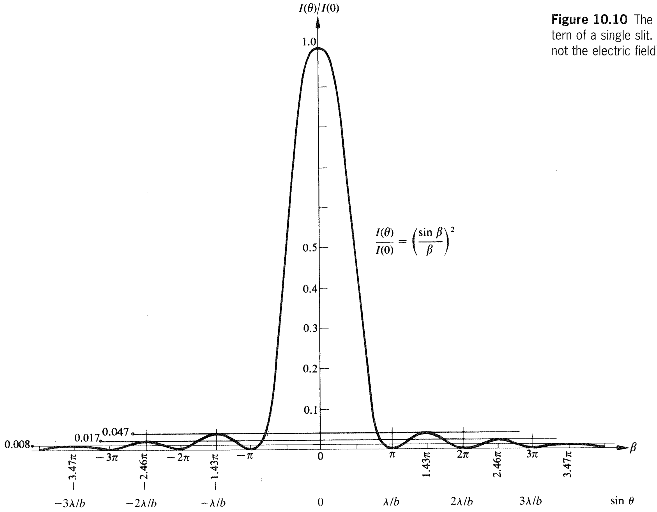


Figure 10.10 The Fraunhofer diffraction pattern of a single slit. This is the irradiance (and not the electric field) distribution

Chapter 5: Diffraction

- **Fraunhofer Diffraction from2 Slits**
  - **In this case the E-field is the sum of that from each slit:**

$$E = C\int_{-b/2}^{b/2} F(z)\,dz + C\int_{a-b/2}^{a+b/2} F(z)\,dz \quad \text{where } F(z) = \sin[\omega t - kR + 2\alpha]$$

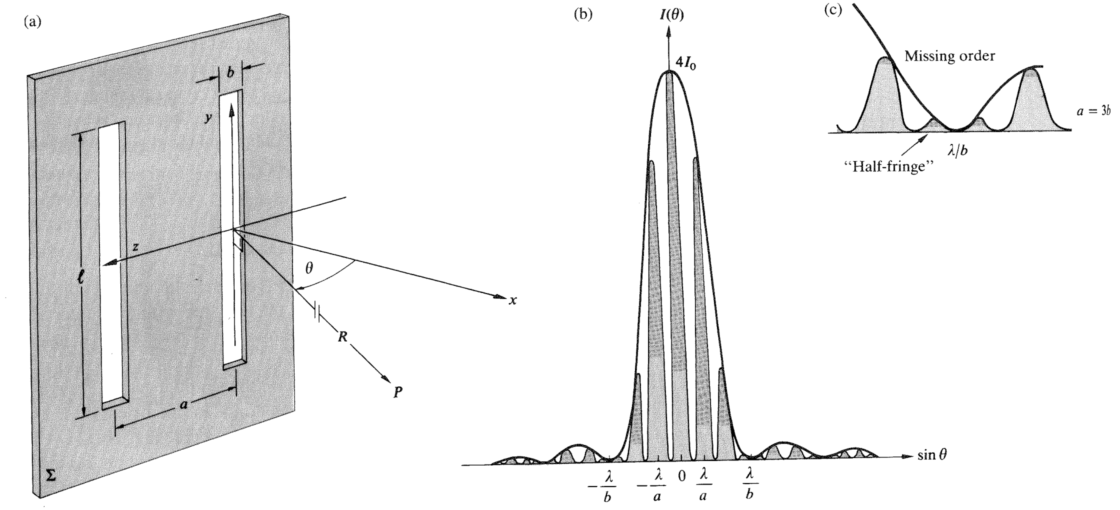
Integration yields:

$$E = bC\left(\frac{\sin\beta}{\beta}\right)[\sin(\omega t - kR) + \sin(\omega t - kR + 2\alpha)] \quad \text{with } \alpha = (ka/2) \text{ and } \beta = (kb/2).$$

When simplified and squared the intensity becomes:

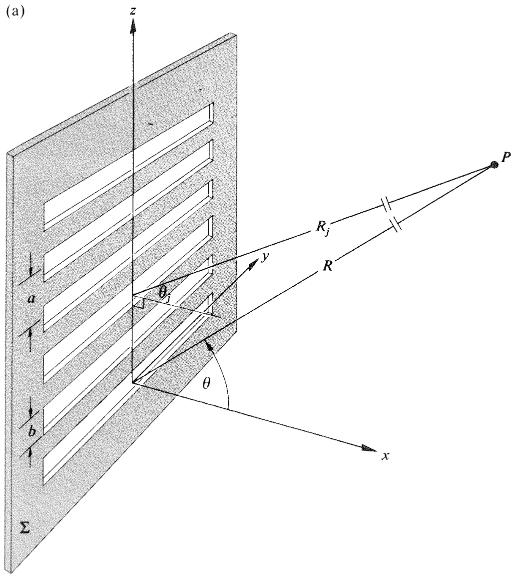
$$I(\theta) = 4I_0\left(\frac{\sin^2\beta}{\beta}\right)\cos^2\alpha$$

Note the modulation of the cos (interference) term by the sinc (diffraction) term.



Chapter 5: Diffraction

- **Diffraction by Many Slits**
  - **Now generalize to N slits:**



$$E = C\int_{-b/2}^{b/2} F(z)dz + C\int_{a-b/2}^{a+b/2} F(z)dz + C\int_{2a-b/2}^{2a+b/2} F(z)dz + C\int_{3a-b/2}^{3a+b/2} F(z)dz$$

$+\ldots+C\int_{Na-b/2}^{Na+b/2} F(z)dz$ with the approximation $r \cong R - z\sin\theta$ the j-th term is:

$$E_j = \frac{C}{k\sin\theta}[\sin(\omega t - kR)\sin(kz\sin\theta) - \cos(\omega t - kR)\cos(kz\sin\theta)]_{ja-b2}^{ja+b/2}$$

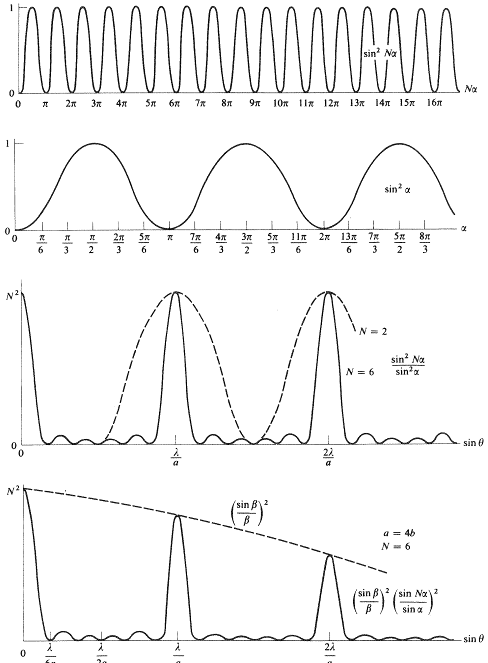
which can be sinplified to:

$E_j = bC\left(\frac{\sin\beta}{\beta}\right)\sin(\omega t - kR + 2\alpha j)$ and upon evaluation of the geometric series (as before):

$E = bC\left(\frac{\sin\beta}{\beta}\right)\left(\frac{\sin N\alpha}{\sin\alpha}\right)\sin[\omega t - kR + (N-1)\alpha]$ and so the intensity is:

$I(\theta) = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$ with maxima at $\alpha = 0,\ \pm\pi, \pm 2\pi, \ldots$

Note from the figure that as N increases the individual maxima get brighter and more distinct.



Chapter 5: Diffraction

- **Diffraction Grating**
  - For a slit separation of a the location of each order is:
  - a $\sin\theta_m = m\lambda$ (note λ dependence)

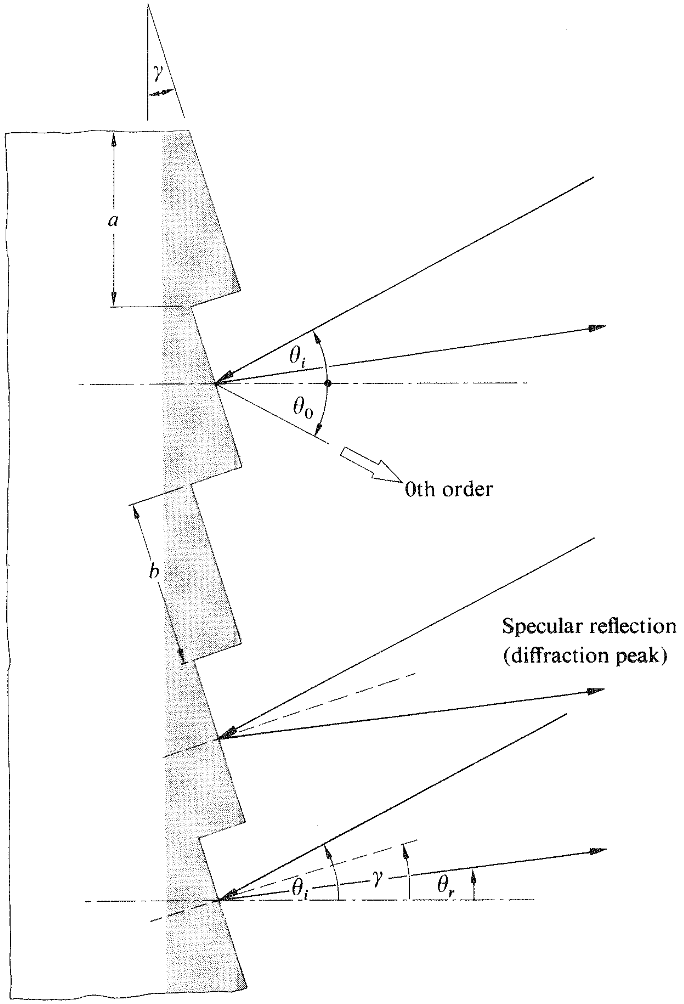
    Thus white-light produces a spectrum at each order (m).

    The angular dispersion can be computed via differentiation.

    A diamond is used to cut groves into glass and aluminized to act as little mirrors.

    It can be used in reflection without aluminizing
  - Grating spectroscopy:
    - A collimating lens can be used to produce an input beam with θ_i = constant. The grating equation:

      $a(\sin\theta_m - \sin\theta_i) = m\lambda$
    - The dispersed light from the grating can be imaged using a lens acting as a camera. Thus the camera "sees" light entering at different field angles according to wavelength. Resulting image is a series of "slit images" displaced according to wavelength.
    - See text for application examples.



Chapter 5: Diffraction & Spectroscopy

- **Diffraction Grating**
  - **Consider the Grating Equation:**

$$m\lambda = a\sin\theta_m$$

Recall the image scale for a lens is:

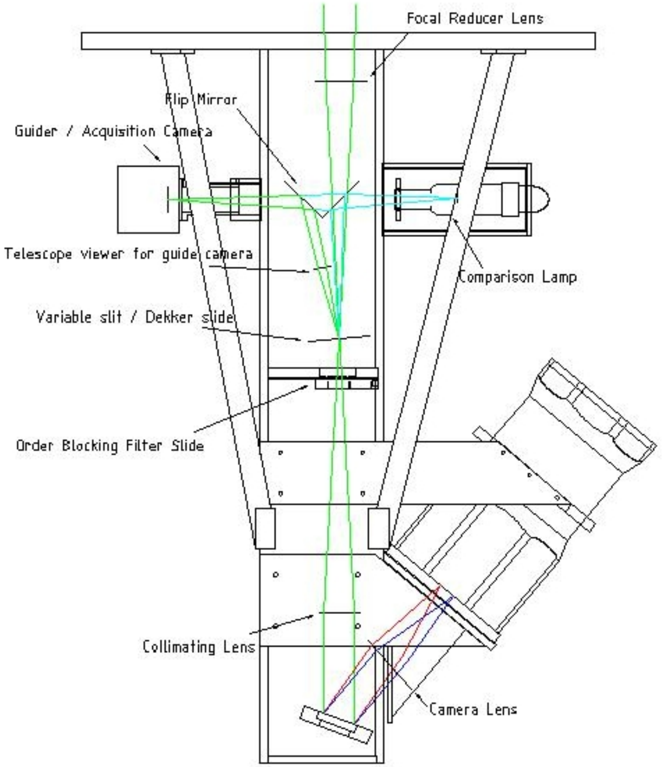
$s = f\theta$ where θ is the image angular extent, s is its physical extent in the image and f is the focal length.

Differentiating both:

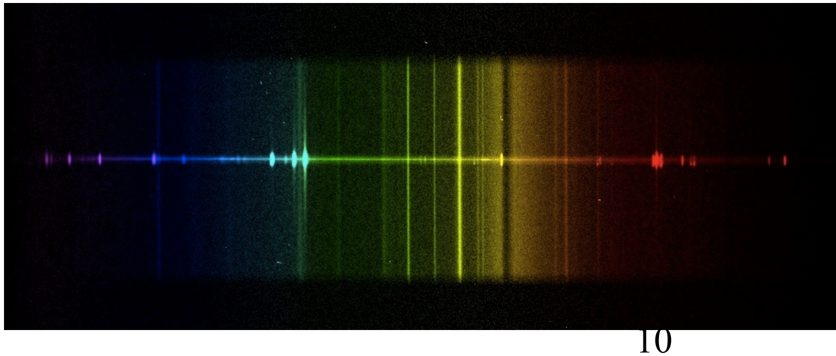
$md\lambda = a\cos\theta_m d\theta$ and $ds = fd\theta$ and so combining:

$$md\lambda = a\cos\theta_m \frac{ds}{f} \quad \text{or:}$$

$$\frac{d\lambda}{ds} = \frac{a\cos\theta_m}{mf}$$ which is called the "spectral dispersion" (nm/μm)



  - **Grating spectroscopy:**
    - **A collimating lens can be used to produce an input beam with θ_i = constant. The role of the camera is to form an image that is a series of "slit images" displaced according to wavelength.**
    - **Taking a calibration spectrum of a source of known spectral lines provides the "wavelength calibration" of position vs. wavelength.**
    - **Spectrum of unknown source then calibrated and features identified.**



Chapter 5: Diffraction

- **Diffraction from a Square Aperture**
  - **Similar to a single slit but we now integrate in 2-d**

Using complex notation the disturbance at P is given by expressing the contribution from each differential element (as before) but now integrating over two dimensions. Fortunately, we can just split the integral into two parts, adding the effect of of the wavelets in the vertical and hroizontal directions. Specifically the differential disturbance produced by distribution of wavelets over a surface (S), with ε_A as the source strength per unit area, is :

$$dE = \left(\frac{\varepsilon_A}{\mathrm{r}}\right) e^{i(\omega t - kr)} dS \quad \text{(the general case)}$$

We next approximate r as :

$r = R[1 - 2(Yy + Zz)/R^2]$ (see text pg. 464 for a justification). We now consider the specific case of a retangular aperture (see figure). Thus, substituting and factoring out the R - term in common :

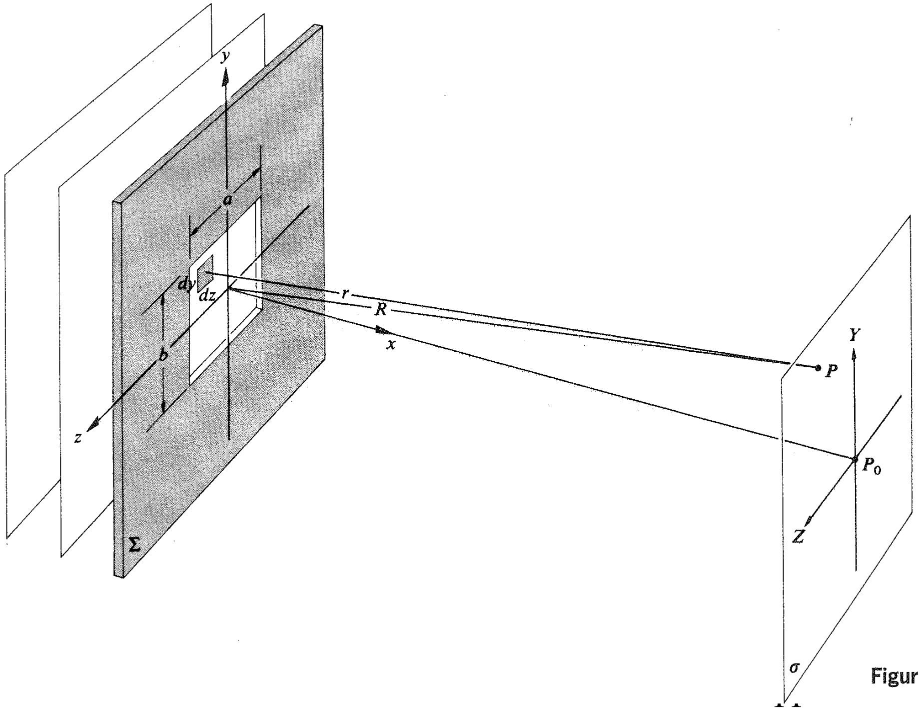
$$E = \frac{\varepsilon_A e^{i(\omega t - kR)}}{R} \int_{-b/2}^{b/2} e^{ikYy/R} dy \int_{-a/2}^{a/2} e^{ikZz/R} dz$$

If we let $\beta' = kbY/2R$ and $\alpha' = kaZ/2R$ we have :

$$E = \frac{\varepsilon_A e^{i(\omega t - kR)}}{R} \left(\frac{\sin\alpha'}{\alpha'}\right)\left(\frac{\sin\beta'}{\beta'}\right) \quad \text{and thus the intensity becomes :}$$

$$I(\theta) = I_0 \left(\frac{\sin\alpha'}{\alpha'}\right)^2 \left(\frac{\sin\beta'}{\beta'}\right)^2$$

Notice that the form of the intensity pattern is the product of two sinc functions, one in each dimension.



Figur

Chapter 5: Diffraction

- **Diffraction from a Circular Aperture**
  - **Similar but we integrate over the aperture in azimuth angle and r.**

Similar to the single slit or retangular aperture we integrate the differential, complex form for the disturbance at P but here we use spherical coordinates :

$$E = \frac{\varepsilon_A e^{i(\omega t - kR)}}{R} \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} e^{i(k\rho q/R)\cos(\phi-\Phi)} \rho d\rho d\phi$$

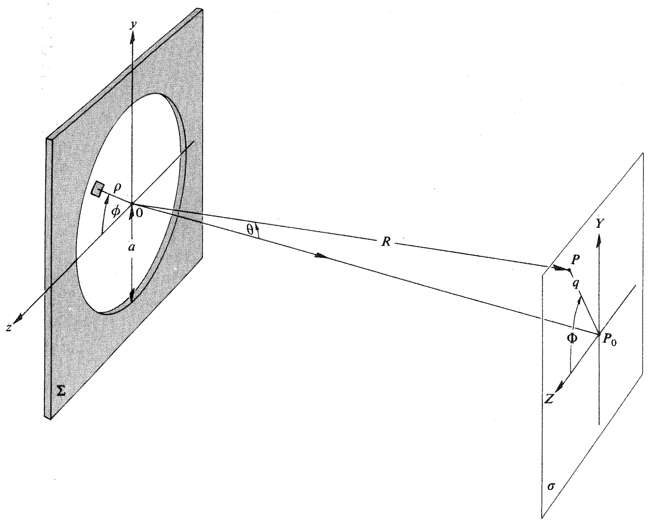
The integral over ϕ is known as the Bessel Function (J_n) and can be evaluated numerically via series expansion. See any textbook on differential equations as they are a well - known solution where axial symmetry is involved. Specifically :

$$J_m(u) = \frac{i^{-m}}{2\pi} \int_0^{2\pi} e^{iu\cos v} dv$$ and so the solution :

$$E = \frac{\varepsilon_A e^{i(\omega t - kR)}}{R} 2\pi a^2 (R/kaq) J_1(kaq/R)$$ and thus :

$$I(\theta) = \frac{2\varepsilon^2{}_A A^2}{R^2} \left[\frac{J_1(kaq/R)}{kaq/R} \right]^2$$ or :

$$I(\theta) = I_0 \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta} \right]^2$$



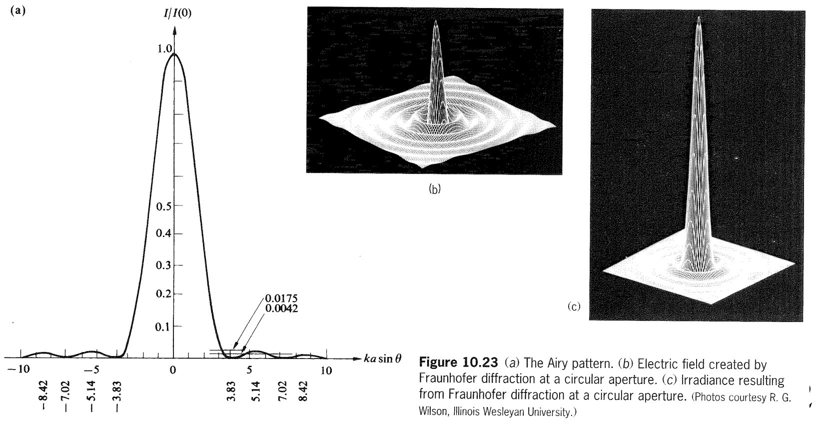
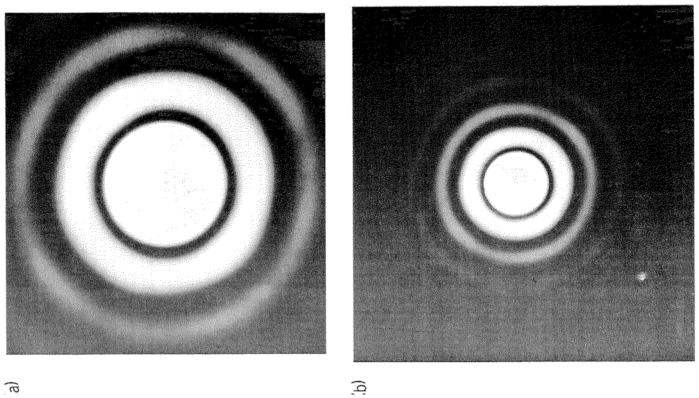


Figure 10.23 *(a)* The Airy pattern. *(b)* Electric field created by Fraunhofer diffraction at a circular aperture. *(c)* Irradiance resulting from Fraunhofer diffraction at a circular aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

Chapter 5: Diffraction

- Implications of Diffraction in Optical Systems
  - Diffraction Limits the Resolution of Optical Systems
    - The larger the aperture the smaller the core and the "Airy Rings"
    - The larger the wavelength the larger the core and the "Airy Rings"
    - Regardless of magnification the resolution of given aperture is limited.
      - Difficult for Hubble to see planets around nearby stars.

$$\theta_{min} = 1.22\ \lambda/D$$



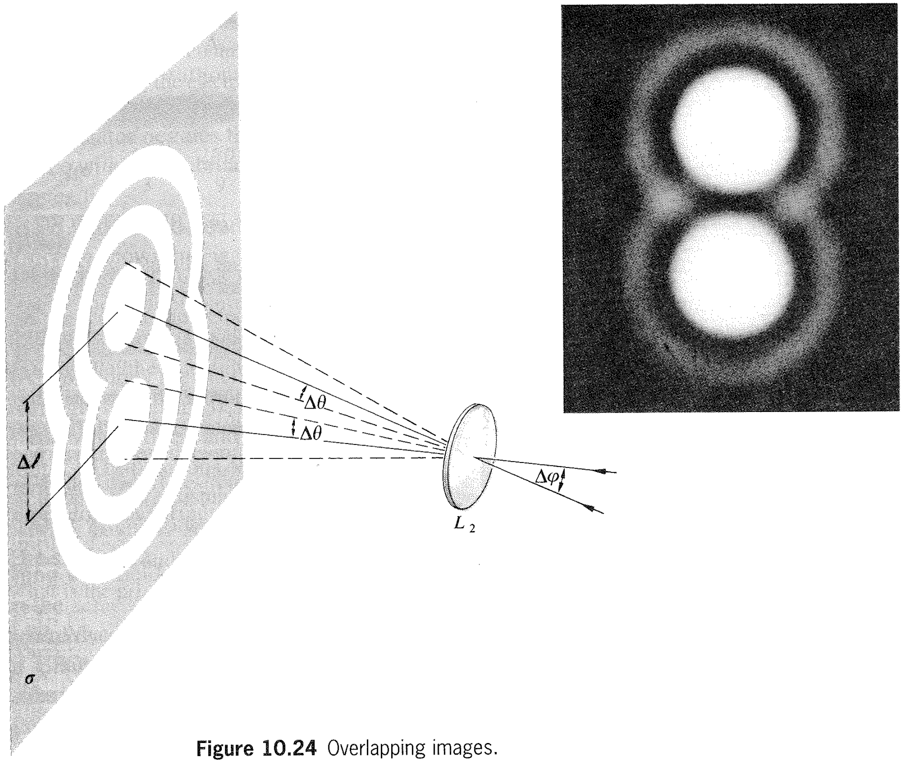


Figure 10.24 Overlapping images.

Chapter 5: Diffraction

- **Fresnel Diffraction**
  - **When either the screen or the source is at a finite distance plane waves are insufficient and Huygens wavelets are located along a curved surface. The math is much more complicated since we must specify the directionality of the secondary wavelets from the source itself [K(θ) = ½(1+cosθ)]. The secondary wavelets around an annulus are in phase relative to the source and hence the phase at P is ωt-k(ρ+r).**

Thus the disturbance at point P for a source strength ε_A will be :

$$dE = K\frac{\varepsilon_A}{r}\cos[\omega t - k(\rho + r)]dS$$

Consider the skinny triangle formed by the annulus and rays to point P.

The law of cosines gives :

$r^2 = \rho^2 + (\rho + r_0)^2 - 2\rho(\rho + r_0)\cos\varphi$ which we differentiate to give :

$2rdr = 2\rho(\rho + r_0)\sin\varphi d\varphi$ but since the area of annulus is :

$dS = \rho d\varphi 2\pi(\rho\sin\varphi)$ we can substitute for $d\varphi$:

$dS = 2\pi\frac{\rho}{(\rho + r_0)}rdr$ and so the disturbance from the l - th annulus is :

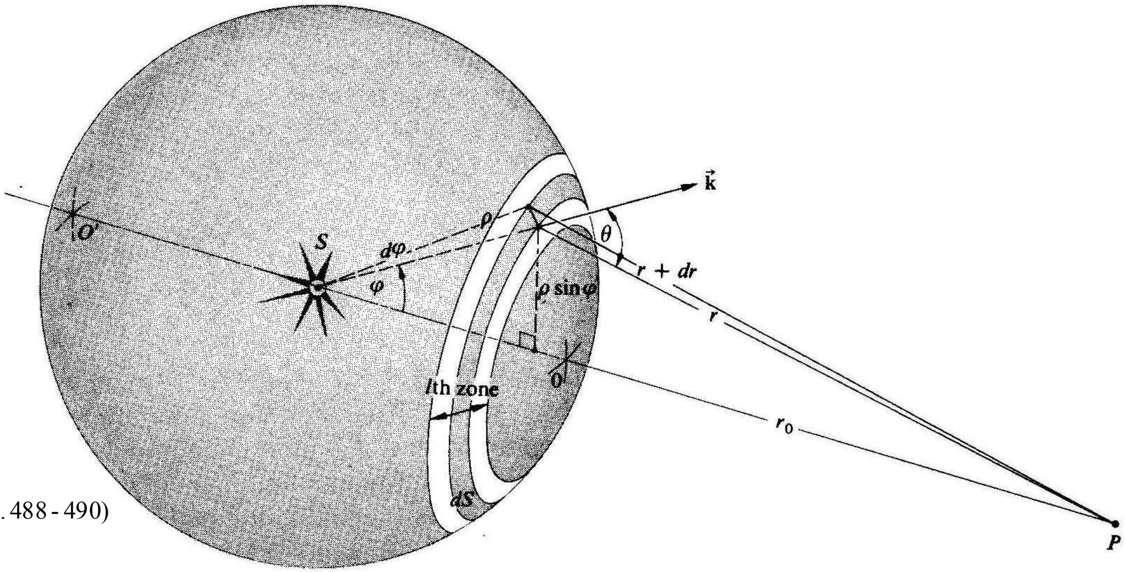
$E_l = K_l 2\pi\frac{\varepsilon_A\rho}{(\rho + r_0)}\int_{r_{l-1}}^{r_l}\cos[\omega t - k(\rho + r_0)]dr$ which is :

$$E_l = \frac{-K_l\varepsilon_A\rho\lambda}{(\rho + r_0)}[\sin(\omega t - k\rho - kr)]_{r=r_{l-1}}^{r=r_l}$$

Since $r_{l-1} = \mathrm{r}_0 + (\mathrm{l}-1)\lambda/2$ and $r_l = \mathrm{r}_0 + \mathrm{l}\lambda/2$ this reduces to :

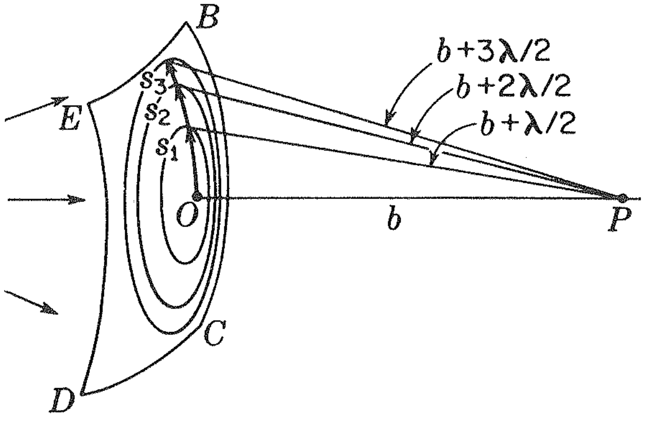
$$E_l = (-1)^{l+1}\frac{2K_l\varepsilon_A\rho\lambda}{(\rho + r_0)}[\sin(\omega t - k(\rho + r_0))]$$

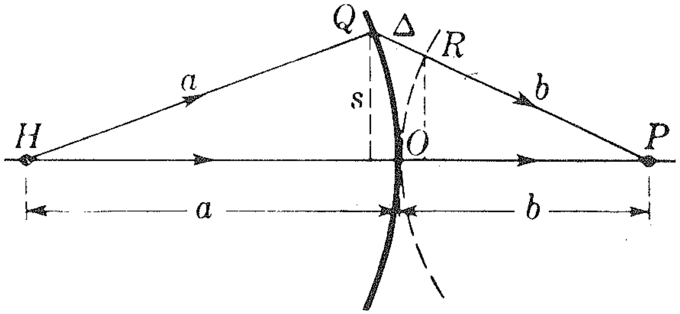
A geometric series or phasor addition can be used to compute the result (pg. 488 - 490)

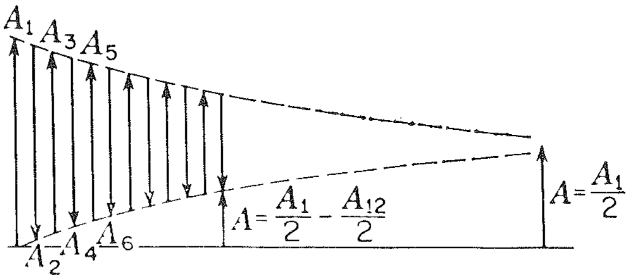


Fresnel's Half-Period Zones

- **Fresnel's Approach**
  - Consider a series of zones s_1, s_2, s_3 … around point O, each λ/2 further from point P
  - Consider phase difference D







For small angles we have:

$$\Delta = \frac{s^2}{2a} + \frac{s^2}{2b} = s\frac{a+b}{2ab} \text{ and}$$

$$m\frac{\lambda}{2} = s_m^2\frac{a+b}{2ab} \text{ so area and intensity are:}$$

$$S_m = \pi(s_m^2 - s_{m-1}^2) \text{ and so}$$

$$S_m = A_m = \pi\frac{\lambda}{2}\frac{2ab}{a+b} = \frac{a}{a+b}\pi b\lambda$$

But every half - period the amplitude inverts:

$$A = A_1 - A_2 + A_3 - A_4 + \ldots + (-1)^{m-1}A_m$$

and amplitude decreases with distance (d_m):

$$A_m = C\frac{S_m}{d_m}(1+\cos\theta)$$

Expanding series and grouping yields:

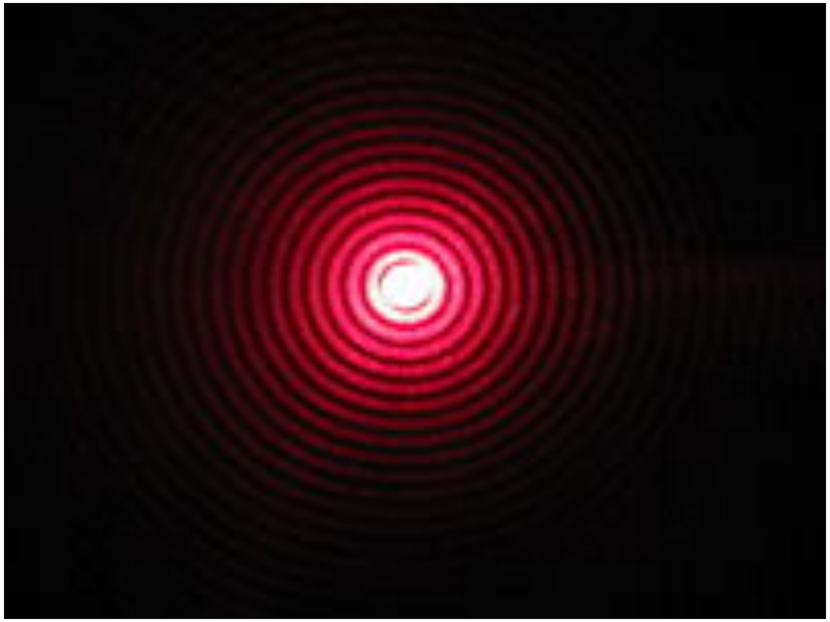
$$A = \frac{A_1}{2} + \frac{A_m}{2} \text{ (if m is odd) and}$$

$$A = \frac{A_1}{2} - \frac{A_m}{2} \text{ (if m is even)}$$

Fresnel Diffraction from a Circular Aperture

- **Imagine zones within a circular aperture**
  - **If radius corresponds to the outer edge of first half-period zone: A = 2A_{inf} and intensity is 4x higher.**
  - **If radius corresponds to the outer edge of second half-period zone:**

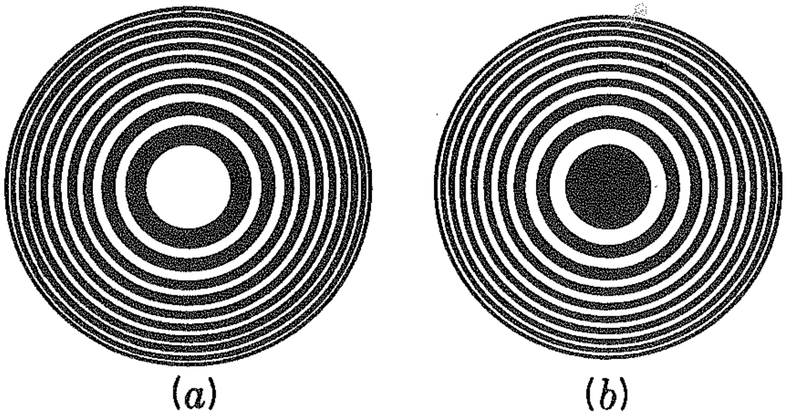
    $A = A_1 - A_2 = 0$!
  - **Intensity drops even though hole is bigger!**
  - **Increasing hole size further results in periodic maxima and minima.**



Fresnel Zone Plates

- **Alternately blocking either the even or odd zones using a mask (right).**
  - **Configured for a specific source.**
- **Result is a lens that will image a distant source!**
  - **"Focal length" will be:**

$$f = \frac{s_m^2}{m\lambda} = \frac{s_1^2}{\lambda}$$



Fresnel Diffraction from a Circular Aperture

- **Now let's add the amplitudes from small circular zones within an aperture**
- **Divide each half period zone into 8 subzones:**

$$d = b + \frac{1}{8}\frac{\lambda}{2},\; b + \frac{2}{8}\frac{\lambda}{2},\; \ldots\; b + \frac{\lambda}{2}$$

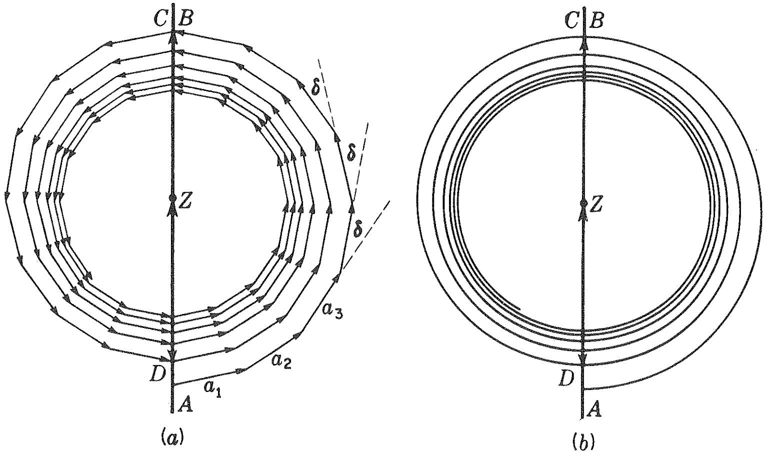
First amplitude is a_1 at right, then add a_2, etc.

After 8 subzones we get vector AB

Repeat for the 8 subzones in next half - period and the result is vector CD and when added produces vector AD (almost zero amplitude).

Adding successive zones produces left hand figure. The "vibration curve" for a circular aperture.

With infinitesimal subzones we get the right hand figure.



(a) (b)

Fresnel Diffraction from a Single Slit

- **Now consider a slit.**
  - **Half-period zones on cylindrical wavefront:**

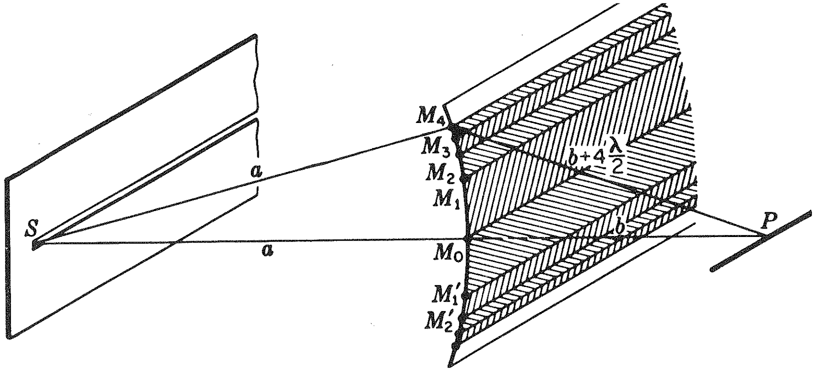
$$b,\ b+\frac{\lambda}{2},\ b+\frac{2\lambda}{2},\ \ldots$$

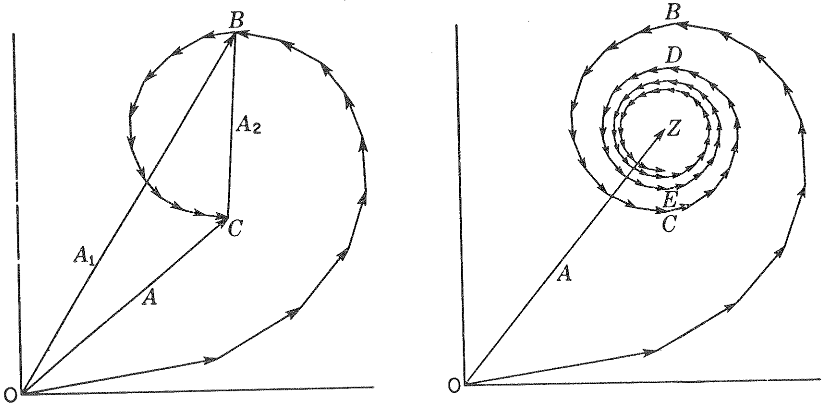
Strip dividion of the wavefront. Similar to circular case but now area is proportional to width so larger obliquity factor. Divission into 9 parts now yields half - period and $\mathrm{A}_1 = \mathrm{OB}$. Second strip yields $\mathrm{A}_2 = \mathrm{BC}$ with $\mathrm{A}_2 < A_1$. Continuing results in the spiral converging to OZ. Infinitesimal strips produce the right hand figure. Vector is a phasor. The addition of the amplitudes from the lower half of the slit also produces a phase lag and a similar but inverted spiral. Quantitatively:

$$\delta = \frac{2\pi}{\lambda}\Delta = \frac{\pi(a+b)}{ab\lambda}s = \frac{\pi}{2}v^2$$

Where we introduce a new variable:

$$v = s\sqrt{\frac{2(a+b)}{ab\lambda}}$$





(a)

(b)

Chapter 5: Fresnel Integrals

- **Fresnel Integrals**
  - **Derivation of Fresnel integrals:**

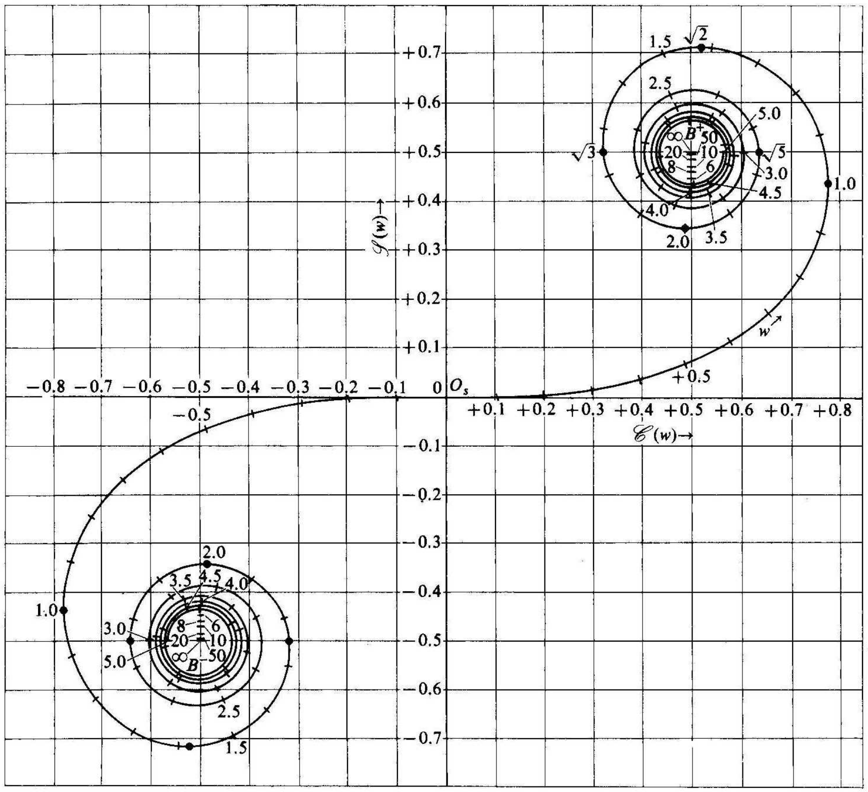
For the X and y coords. on Cornu's sprial :

$$dx = d\nu \cos\delta = \cos\frac{\pi\nu^2}{2}d\nu \text{ and}$$

$$dy = d\nu \sin\delta = \sin\frac{\pi\nu^2}{2}d\nu \text{ and so :}$$

$$x = \int_0^\nu \cos\frac{\pi\nu^2}{2}d\nu \text{ and } y = \int_0^\nu \sin\frac{\pi\nu^2}{2}d\nu$$

They cannot be integrated in closed form but can be numerically evaluated.



Chapter 5: Fresnel Diffraction from Edge

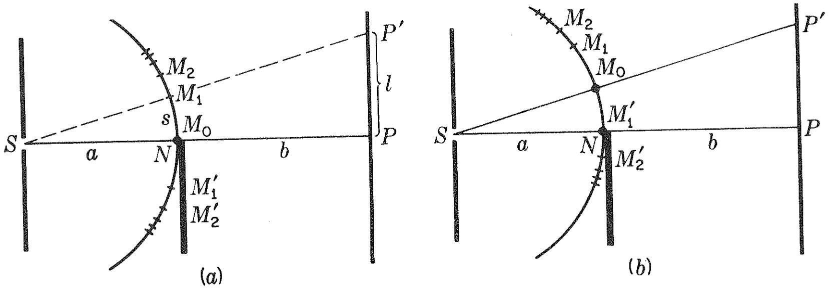
- **Fresnel Diffraction from Edge**
  - **Start at point P where the amplitude is OZ**
  - **Along the screen the verctor head remains fixed an the tail moves along the spiral to a maximum at b'.**
  - **Continuing along the screen the amplitude goes through B' and a minimum at c'.**
  - **Secondary maxima (fringes) occur at d'.**
  - **Ampiltude decreases and converges to OZ'.**
  - **Scale of the pattern:**

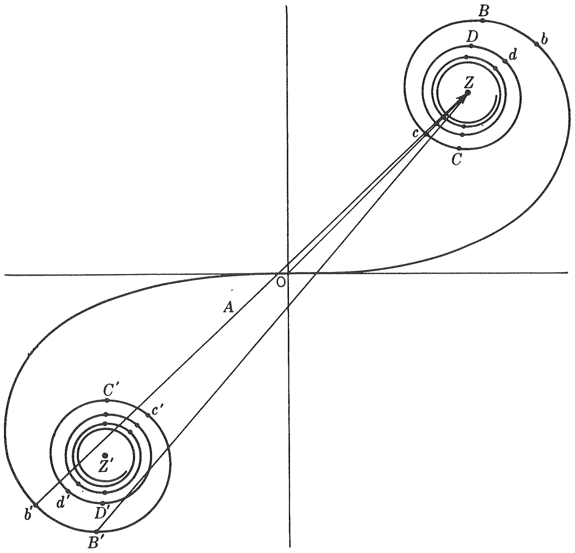
Let distances a = b = 100 cm, and = 500 nm..

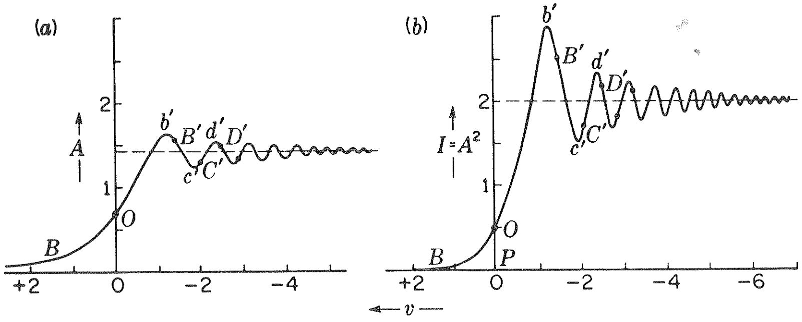
So the distance along the wavefront is :

$$s = v\sqrt{\frac{ab\lambda}{2(a+b)}} = 0.0354v \quad \text{and along the screen :}$$

$$l = \frac{a+b}{a}s = v\sqrt{\frac{b\lambda(a+b)}{2a}} = 0.0708v \text{ cm}$$

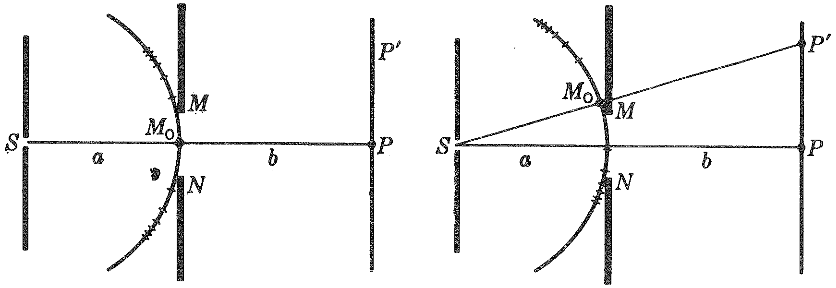




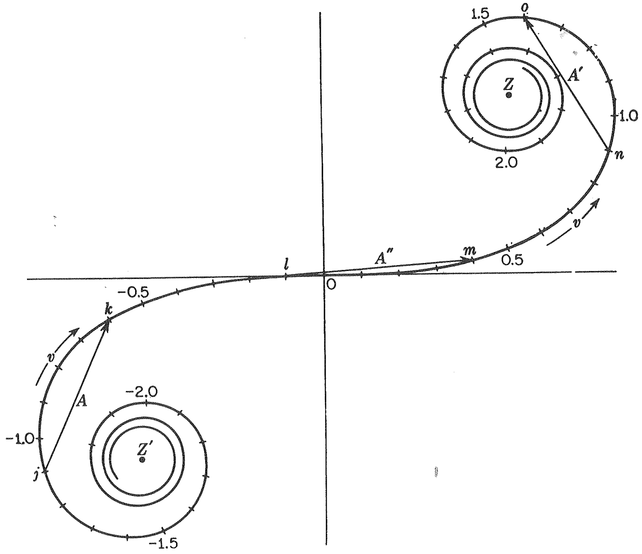


Chapter 5: Fresnel Diffraction from Slit

- **Fresnel Diffraction from Slit**
  - **Procedure is similar but each side acts as an opaque edge.**
  - **For a = 100cm, b = 400cm, λ = 400 nm, and Δs = 0.02 cm, Δv = 0.5**
  - **Intensity at P' found by drawing vector Δv = 0.5 at different positions along the spiral and measuring the corresponding amplitude (A and A')**

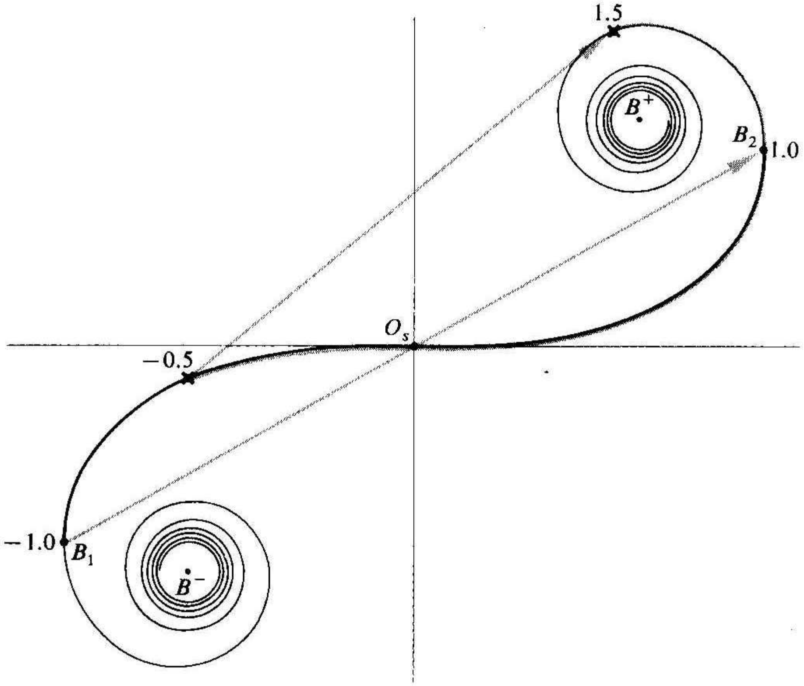


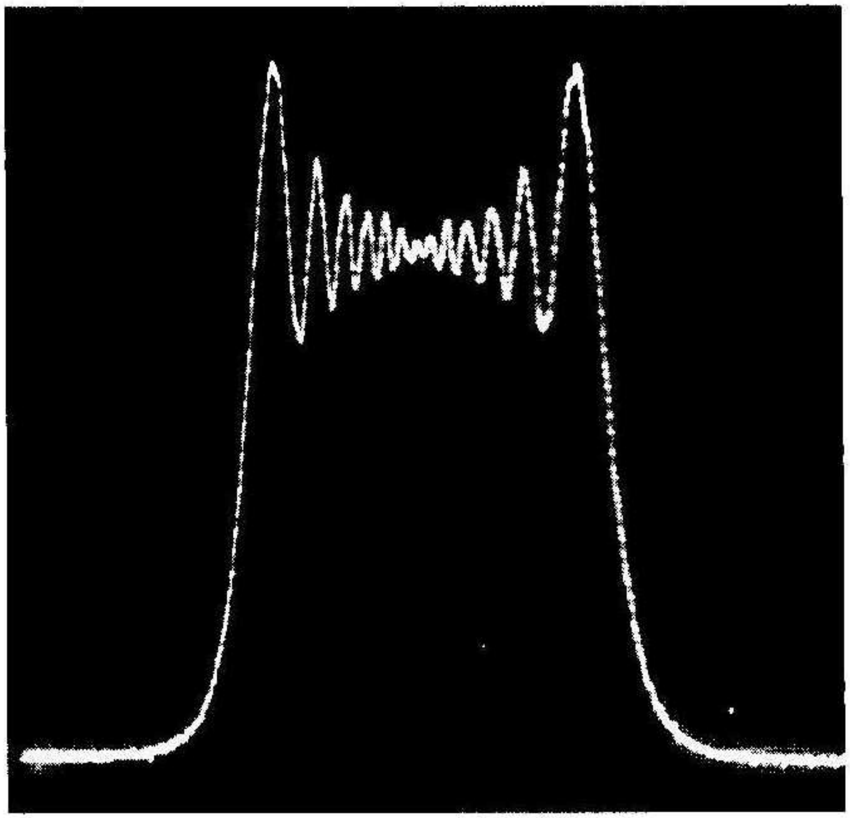
(a) (b)



Chapter 5: Fresnel Diffraction

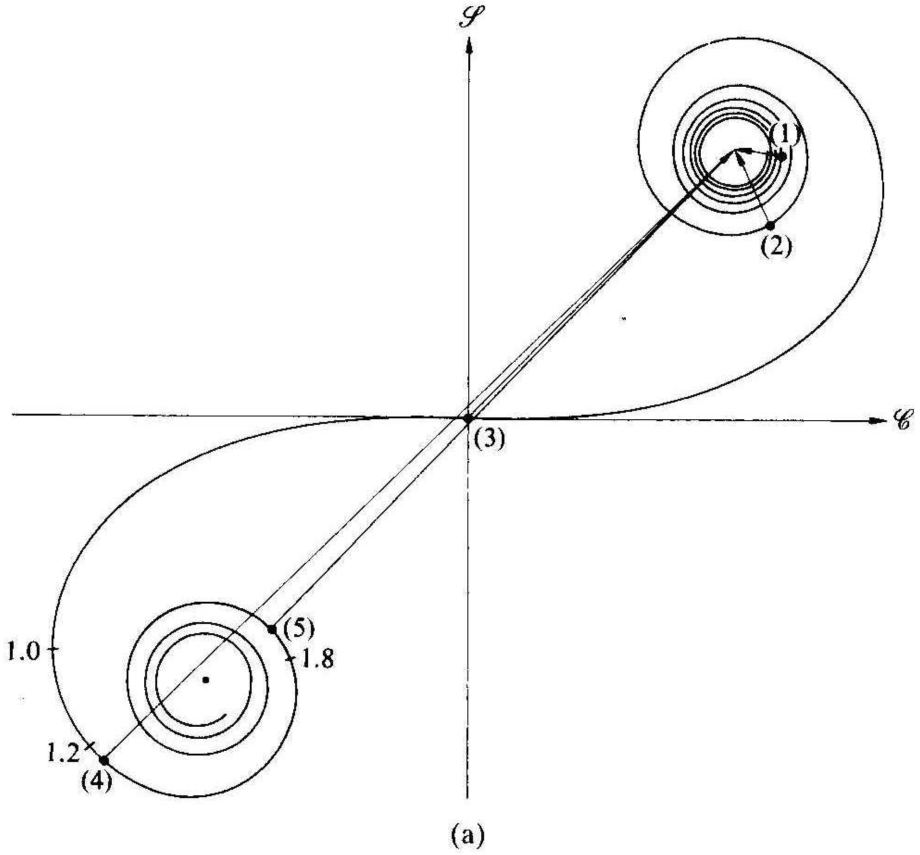
- **Fresnel Diffraction from Rectangular (slit) Aperture**
  - See Hecht sec. 10.3.8 for alternative explanation of how to use the spiral to get the complex amplitude given the silt width (positions on the spiral).

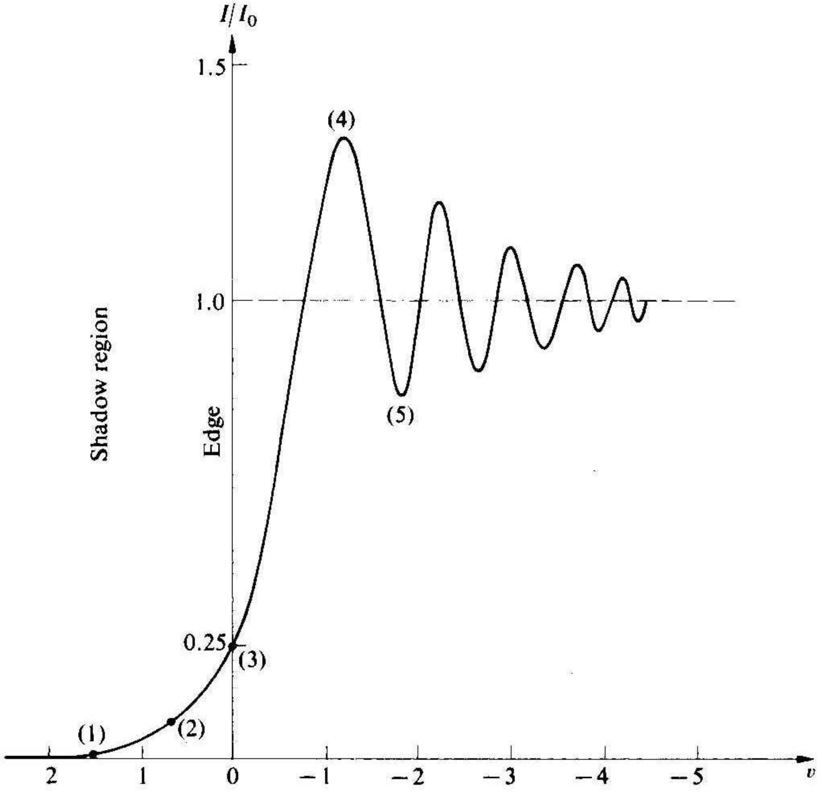




Chapter 5: Fresnel Diffraction

- **Fresnel Diffraction from Semi-opaque Aperture**





Homework this Week

No HW this week. Study for the final.