Phys 2310 Fri. Nov. 7, 2014
Today's Topics

- **Begin Chapter 15: The Superposition of Waves**
- **Reading for Next Time**

Reading this Week

By Friday:

- **Begin Ch. 15 (15.1 – 15.3) Addition of Waves of the Same Frequency, Addition of Waves of Different Frequency,**
- **Read Supplementary Material:**
  - **Anharmonic Periodic Waves**

Homework this Week

Chapter 15: #1 (use Excel), 3, 6, 10, 33

Due Wed. Nov. 16

Chapter 15: Addition with Same Frequency

- **Algebraic Method**
  - **Recall that light waves are not affected when the beams cross.**
  - **The disturbance (E-field amplitude) just adds locally.**

Consider two waves:

$E_1 = E_{01}\sin(\omega t + \alpha_1)$ and $E_2 = E_{02}\sin(\omega t + \alpha_2)$

where $\alpha = -(kx + \varepsilon)$ is the spatial portion of the phase.

Adding the two waves (trig. identities):

$E = E_{01}(\sin\omega t\cos\alpha_1 + \cos\omega t\sin\alpha_1) + E_{02}(\sin\omega t\cos\alpha_2 + \cos\omega t\sin\alpha_2)$

and so when the spatial and time-dependent portions are separated:

$\mathrm{E} = (E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2)\sin\omega t + (E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2)\cos\omega t$

Now note that the terms in () are constant in time. So we define:

$E_0\cos\alpha = E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2$ and $E_0\sin\alpha = E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2$

Squaring and adding to get the intensity:

$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_2 - \alpha_1)$ (note interference term)

Dividing to get the phase:

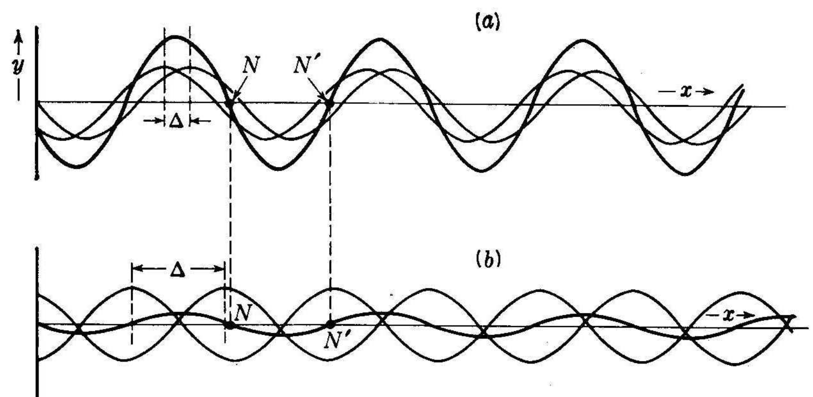
$\tan\alpha = \dfrac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2}$ thus:

$E = E_0\cos\alpha\sin\omega t + E_0\sin\alpha\cos\omega t$

Note the interference term varies with OPL:

$\delta = (kx_1 + \varepsilon_1) - (kx_2 + \varepsilon_2)$ or:

$\delta = \dfrac{2\pi}{\lambda_0} n(x_1 - x_2)$ (where $n(x_1 - x_2)$ is the OPL)



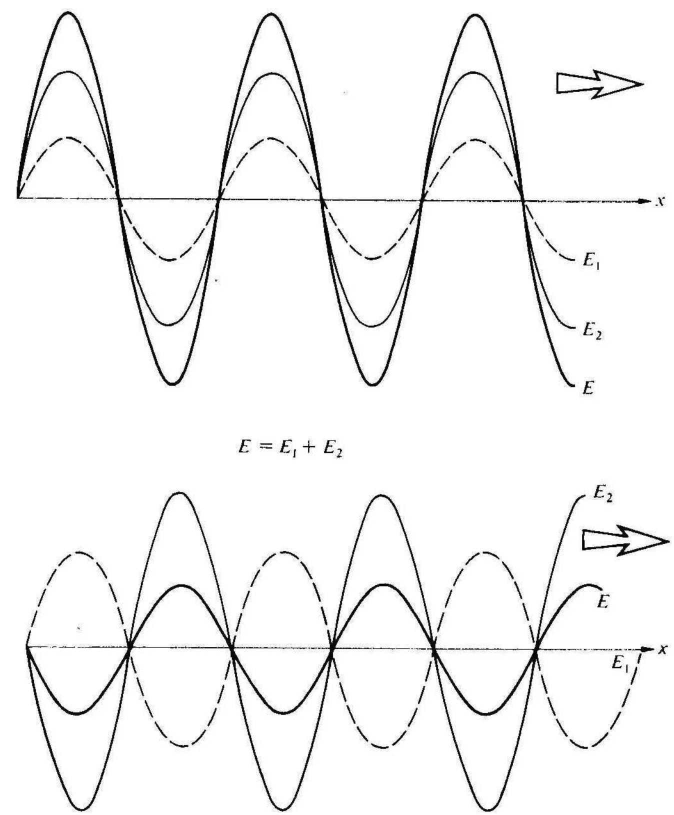
Chapter 15: Addition with Same Frequency

- **Superposition of Many Waves**
  - **Now consider that case of many waves:**

$$E = \sum_{i=1}^{N} E_{0i}\cos(\alpha_i \pm \omega t) = E_0 \cos(\alpha \pm \omega t) \ \text{ where:}$$

$$E_0^{\ 2} = \sum_{i=1}^{N} E_{0i}^{\ 2} + 2\sum_{j>1}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos(\alpha_1 - \alpha_2) \ \text{ and:}$$

$$\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i}\sin\alpha_i}{\sum_{i=1}^{N} E_{0i}\cos\alpha_i}$$



Note that since all have the same frequency the result will also have the same frequency, only the amplitude changes.

Chapter 15: Addition with Same Frequency

- **Complex Method and Phasor Addition**
  - **Recall that Phasor addition is like vector addition in polar coords.**

If $E_1 = E_{01}e^{i(\alpha_1 \mp \omega t)}$ and there are N waves:

$$E = \left[\sum_{j=1}^{N} E_{0i}e^{i\alpha_j}\right]e^{i\omega t} \text{ where}$$

$$E_0 e^{i\alpha} = \left[\sum_{j=1}^{N} E_{0i}e^{i\alpha_j}\right] \text{ is known as the complex amplitude.}$$

Recall that since $E_0^{\ 2} = (E_0e^{i\alpha})(E_0e^{i\alpha})^*$ and so for N = 2:

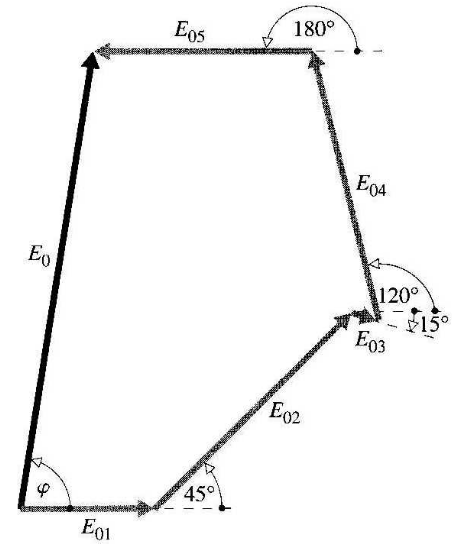
$$E_0^{\ 2} = (E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2})(E_{01}e^{-i\alpha_1} + E_{02}e^{-i\alpha_2})$$

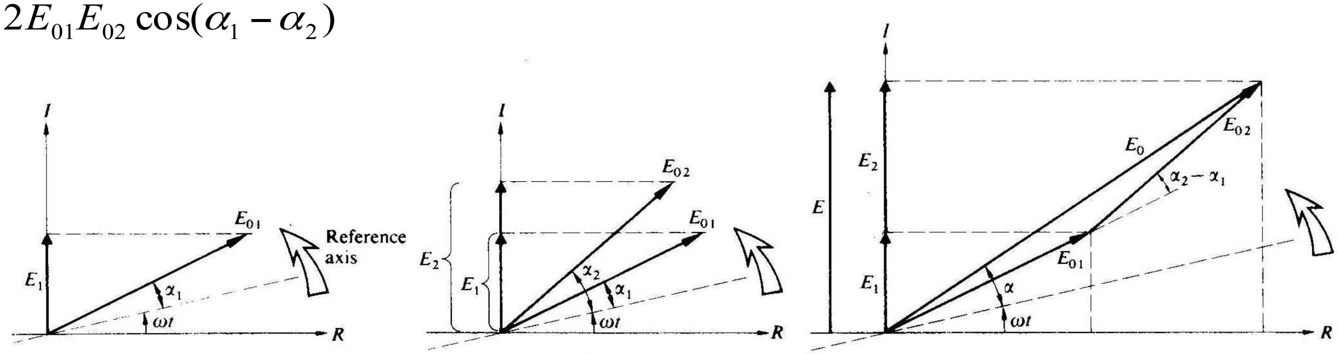
$$E_0^{\ 2} = E_{01}^{\ 2} + E_{02}^{\ 2} + 2E_{01}E_{02}[e^{i(\alpha_1-\alpha_2)} + e^{-i(\alpha_1-\alpha_2)}]$$

$$E_0^{\ 2} = E_{01}^{\ 2} + E_{02}^{\ 2} + 2E_{01}E_{02}\cos(\alpha_1 - \alpha_2)$$

Similarly for Phasor additionthe law of cosines gives:

$$E_0^{\ 2} = E_{01}^{\ 2} + E_{02}^{\ 2} + 2E_{01}E_{02}\cos(\alpha_1 - \alpha_2)$$





(a) (b) (c)

- **Complex Method and Phasor Addition**
  - **Consider the Phasor addition of 5 waves**

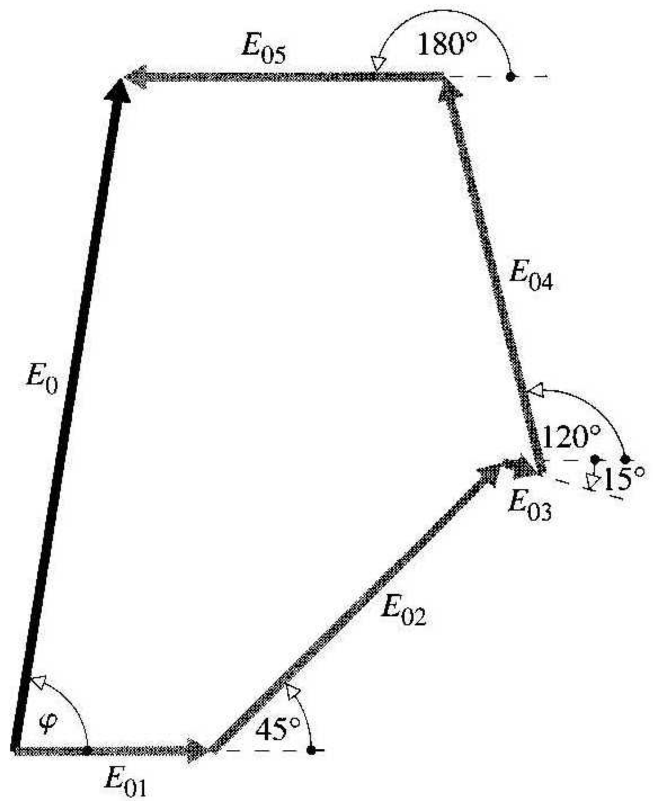
$$E_1 = 5\sin\omega t$$

$$E_2 = 10\sin(\omega t + 45^o)$$

$$E_3 = \sin(\omega t - 15^o)$$

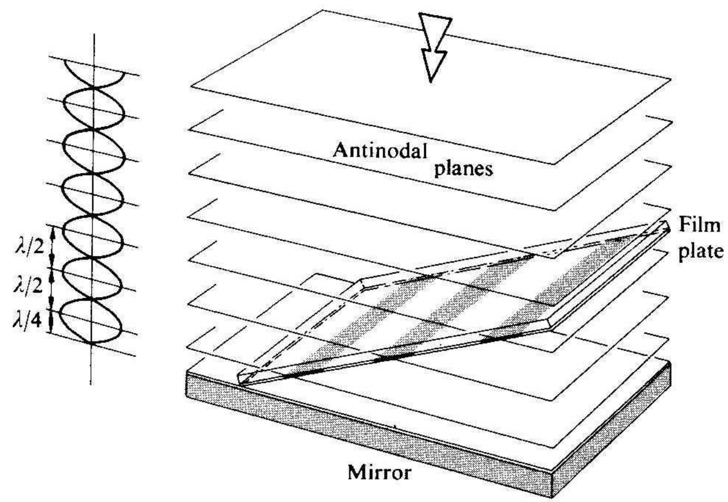
$$E_4 = 10\sin(\omega t + 120^o)$$

$$E_5 = 8\sin(\omega t + 180^o)$$



Chapter 15: Addition with Same Frequency

- **Standing Waves**
  - **Weiner showed that the E field at the surface of a mirror must be zero, just like the E&M theory of a conductor.**
  - **Reflected waves are like standing waves with a <u>phase change of π</u>.**



In order to satisfy this boundary condition consider both waves:

$$E = E_I + E_R = E_{0I}[\sin(kx + \omega t) + \sin(kx - \omega t)]$$

And since:

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)$$ we have:

$$E(x,t) = 2E_{0I}\sin kx\cos\omega t \quad \text{(standing wave)}$$

Note that one part oscillates in time but the amplitude or "envelope" doesn't, hence the name. This results in nodes and anti-nodes.

In addition, Weiner's experiment showed that reflected light can interfere.

This technique can be used to measure the wavelength of light from the OPD.

• Beats from Two Waves of Different Frequency

Consider the addition of two waves of different frequency:

$E_1 = E_{01}\cos(k_1 x - \omega_1 t)$ and $E_2 = E_{02}\cos(k_2 x - \omega_2 t)$

If the amplitudes are the same teir addition yields:

$E = E_{01}[\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)]$ and since:

$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$ we have:

$$E = 2E_{01}\cos\frac{1}{2}[(k_1+k_2)x - (\omega_1+\omega_2)t]\times\cos\frac{1}{2}[(k_1-k_2)x - (\omega_1-\omega_2)t]$$

Note that this wave has a frequency equal to the average of the two but is modulated by a much lower frequency wave $(\omega_1 - \omega_2)$:

If we re-cast the differences in frequency and propagation number:

$\omega_m \equiv \frac{1}{2}(\omega_1 - \omega_2)$ and $\overline{\omega} \equiv \frac{1}{2}(\omega_1 + \omega_2)$ and:

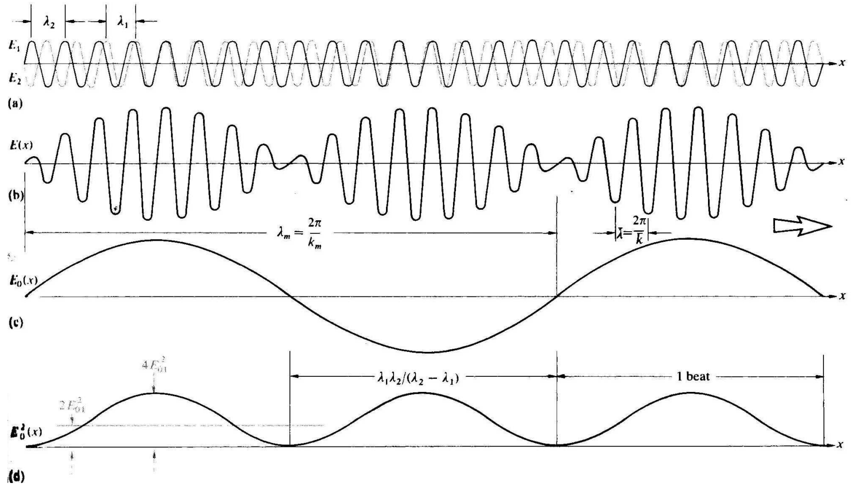
$k_m \equiv \frac{1}{2}(k_1 - k_2)$ and $\overline{k} \equiv \frac{1}{2}(k_1 + k_2)$ we have:

$E = 2E_{01}\cos(k_m x - \omega_m t)\cos(\overline{k}x - \overline{\omega}t)$

Note that the irrandiance becomes:

$E_0{}^2 = 4E_{01}{}^2\cos^2(k_m x - \omega_m t)$ which is:

$E_0{}^2 = 2E_{01}{}^2[1 + \cos(2k_m x - 2\omega_m t)]$



Chapter 15: Anharmonic Periodic Waves

- **Fourier Series**
  - **Recall that we stated that any periodic function can be synthesized by as sum of harmonic waves (sines and cosines). This is Fourier's Theorm. Sines and cosines are used because they are orthogonal, i.e., independent (see text on linear algebra).**

$$f(x) = C_0 + C_1 \cos\left(\frac{2\pi}{\lambda}x + \varepsilon_1\right) + C_2 \cos\left(\frac{2\pi}{\lambda/2}x + \varepsilon_2\right) + \ldots$$

In the figure at right see how 6 harmonic functions can generate the complicated result, and vice versa.

Traditionally we use both cosines and sines since sines are odd and can provide a phase shift too:

If $k = 2\pi/\lambda$ then we can reformulate as:

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mkx + \sum_{m=1}^{\infty} B_m \sin mkx \quad \text{(sum over wavevectors)}$$

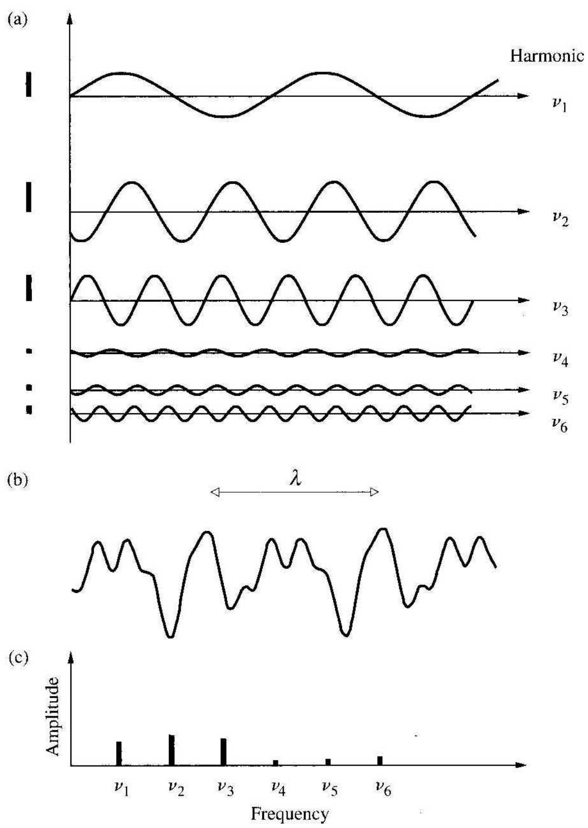
where the amplitudes can be computed by integrating both sides and noting orthogonality:

$$\int_0^\lambda \sin akx \cos bkx\, dx = 0, \quad \int_0^\lambda \cos akx \cos bkx\, dx = \frac{\lambda}{2}\delta_{ab} \quad \text{and} \quad \int_0^\lambda \sin akx \sin bkx\, dx = \frac{\lambda}{2}\delta_{ab}$$

where δ_{ab} is 0 if $a \neq b$ and 1 if $a = b$ (Kronecker delta). Thus we find:

$$A_m = \frac{2}{\lambda}\int_0^\lambda f(x) \cos mkx dx \quad \text{and} \quad B_m = \frac{2}{\lambda}\int_0^\lambda f(x) \sin mkx dx$$

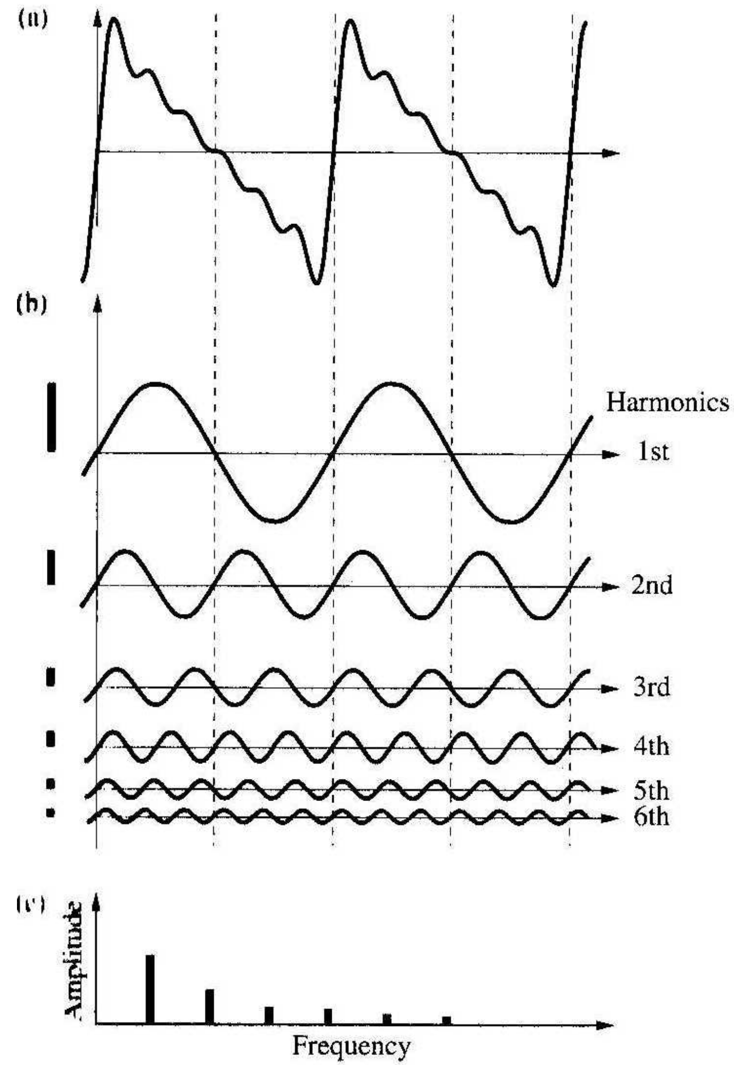
So, given a function we can compute the Fourier amplitudes to model it. Note the symmetry between the function and the amplitudes of the Fourier series. In addition a pure even function: $f(x) = f(-x)$ will contain only cosines and a purely odd function: $f(-x) = -f(x)$ will only have sine terms.



Note that a complex waveform can then be "modeled" just from the Fourier coefficients (A_m, B_m). This greatly reduces a digital file size and is the basis of all compression algorithms. All trade "fidelity" vs. file size. A familiar example is MP3. When we model the waveform, i.e., take a Fourier transform, many times per second (576 for early versions of MP3), we end up with 576 sets (spectra) of Fourier amplitudes each second. The frequency range we sample (spectrum) depends on the application. Lots of software is available for recording and converting audio to MP3.

Chapter 15: Anharmonic Periodic Waves

- Fourier Series
  - Knowing a function, f(x), we can compute the coefficients. The number of terms depends on the precision required.
  - Note how only 6 terms (in sine) can reproduce the sawtooth function rather well since it is odd.
  - Adding more terms is easy with a computer (even Excel).
  - A few hundred terms may be necessary to accurately reproduce both the slopes and the discontinuities. The number required depends on the precision required.
  - Note the decreasing amplitude of the higher-order terms in the frequency spectrum (typical).



Chapter 15: Anharmonic Periodic Waves

- **Fourier Series Example**
  - **Lets follow along with the book with an example square-wave.**

$$f(x)=\begin{cases}+1 \text{ when } 0<x<\lambda/2 \\ -1 \text{ when } \lambda/2<x<\lambda\end{cases} \text{ so using only sines } (A_m=0, \text{ odd function}):$$

$$B_m=\frac{2}{\lambda}\int_0^{\lambda/2}+1\sin mkx\,dx+\frac{2}{\lambda}\int_{\lambda/2}^{\lambda}-1\sin mkx\,dx \text{ and thus :}$$

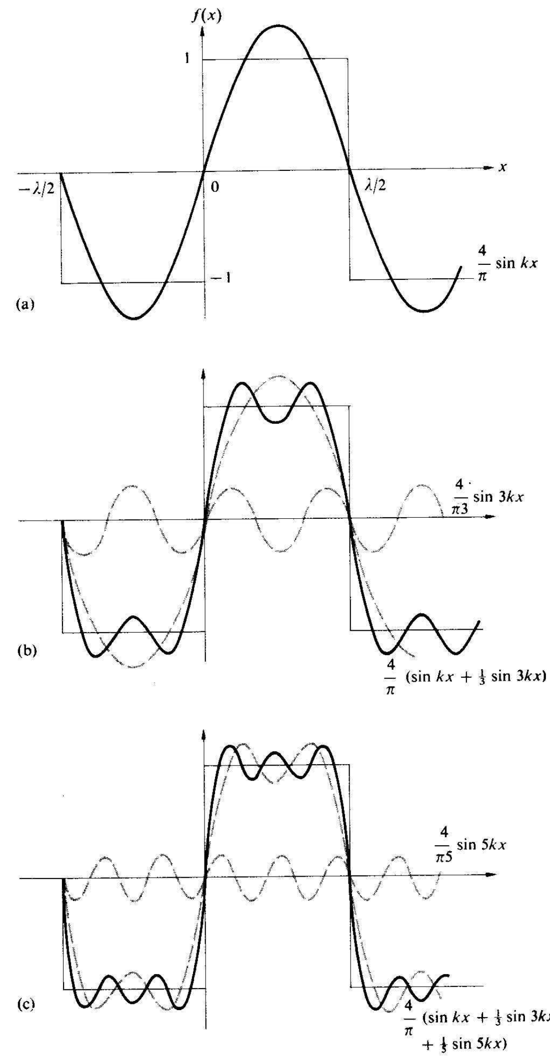
$$B_m=\frac{1}{m\pi}[-\cos mkx]_0^{\lambda/2}+\frac{1}{m\pi}[\cos mkx]_{\lambda/2}^{0} \text{ or :}$$

$$B_m=\frac{2}{m\pi}(1-\cos m\pi) \text{ since k}=2\pi/\lambda. \text{ Thus the Fourier coefficients are :}$$

$$B_1=4/\pi,\ B_2=0,\ B_3=4/3\pi,\ B_4=0, B_4=4/5\pi \text{ and so on. Thus :}$$

$$f(x)=\frac{4}{\pi}\left(\sin kx+\frac{1}{3}\sin 3kx+\frac{1}{5}\sin 5kx+\ldots\right)$$

Note the decreasing amplitude of the higher - order terms.
See Figure 7.21 and text for a description of how as the peak width narrows higer order harmonics are necessary to model the wave. That is, reproducing small features requires more Fourier coefficients (apmplitudes).



Chapter 15: Non-periodic Waves

- **Fourier Integrals**
  - **If we accept that we can model any function with an infinite series of sines and cosines it seems reasonable to generalize Fourier series to Fourier Integrals. See figure 7.21 for how In this case we have the Fourier Transform:**

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos kx\,dk + \int_0^\infty B(k)\sin kx\,dk\right] \text{ with :}$$

$$A(k) = \int_{-\infty}^{\infty} f(x)\cos kx\,dx \text{ and } B(k) = \int_{-\infty}^{\infty} f(x)\sin kx\,dx$$

An interesting Fourier transform pair is the square wave

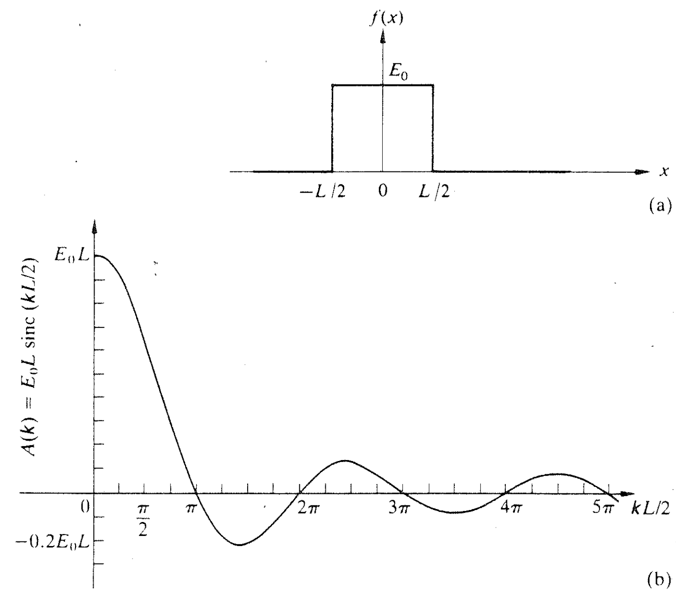
$f(x) = E_0$ when $|x| < L/2$ otherwise 0.

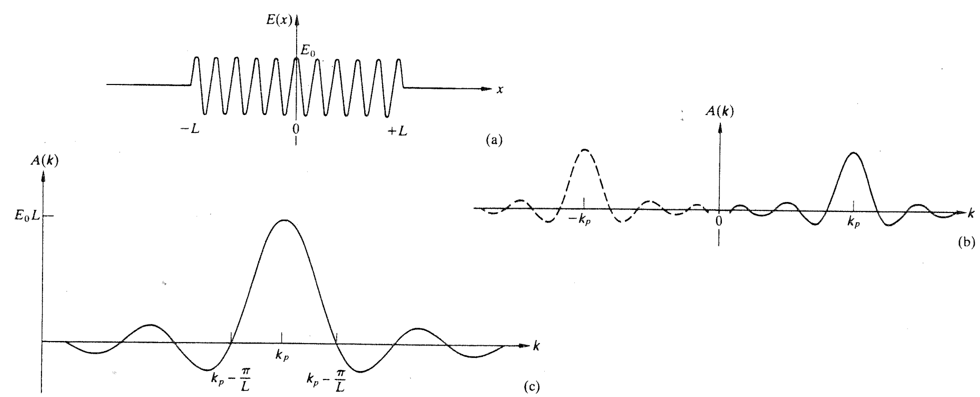
Its transform is the sinc function :

$$A(k) = E_0 L \frac{\sin(kL/2)}{kL/2}$$

Chapter 15: Pulses and Wave Packets

- **Square Pulse**
  - Fourier transform of a square pulse is called the Sinc function
- **Finite Wavetrain**
  - A given Fourier component goes on forever
  - A finite wavetrain requires additional Fourier modes.
    - Think of the wave being modulated by a square pulse.
    - Thus a finite wavetrain (pulse) can be thought of as a wave packet.
    - A the wavetrain becomes long its frequency spectrum shrinks and vice versa.
- **Coherence Length**
  - Any real E-M wave is not absolutely monochomatic but has a natural width.
  - There must be a coherence length and a coherence time as a result.
    - Namely: $\Delta l_c = c\Delta t_c$
  - Coherence length and time are a measure of space and time over which the wave has an approximately constant wavelength or frequency.
  - Note that white light (large bandwidth) can then be understood as the superposition of large numbers of monochromatic waves (many Fourier modes)





Chapter 15: Non-periodic Waves

- **Discrete Fourier Transform**
  - Often our signal consists of a set of discrete points (e.g. from a digital detector)
  - The Fourier integral is approximated by a summation
    - The resulting transform can then be filtered to remove unwanted frequencies (e.g. interference) and then transformed back.
  - This techniques can be further generalized to muti-dimensional data, such as a 2-d images
    - See spatial filtering examples in the textbook.
  - Two excellent references:
    - The Fourier Transform and Its Application (Bracewell)
    - A Student's Guide to Fourier Transforms: with Applications in Physics and Engineering (James)

Reading this Week

By Friday:

Finish Reading Ch. 15 (15.1 – 15.3) Addition of Waves of the Same Frequency, Addition of Waves of Different Frequency Anharmonic Periodic Waves

Homework this Week

Chapter 15:

Due Wed. Nov. 16