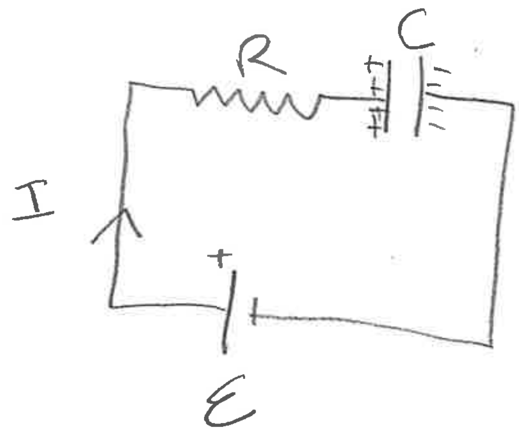
RC Circuit



Charging

- as charge builds on the capacitor, the current in the circuit decreases.

Use loop Rule!

$$\mathcal{E} - IR - V_c = 0$$

$$V_c = \frac{Q}{C}$$

$$\mathcal{E} - IR - \frac{Q}{C} = 0$$

- current in the circuit is the time derivative of charge on the capacitor.

$$I = \frac{dQ}{dt}$$

$$\mathcal{E} - \frac{dQ}{dt}R - \frac{Q}{C} = 0$$

Put Q terms on one side, t terms on the other. (2)

$$\mathcal{E} - \frac{dQ}{dt}R - \frac{Q}{C} = 0$$

$$+\frac{dQ}{dt}R \qquad\qquad +\frac{dQ}{dt}R$$

$$\mathcal{E} - \frac{Q}{C} = \frac{dQ}{dt}R$$

$$C\mathcal{E} - Q = \frac{dQ}{dt}RC$$

$$\frac{1}{RC}\left[C\mathcal{E} - Q\right] = \frac{dQ}{dt}$$

$$\frac{dQ}{C\mathcal{E} - Q} = \frac{dt}{RC}$$

First order, linear, differential equation

$$\int \frac{dQ}{C\mathcal{E} - Q} = \int \frac{dt}{RC}$$

$$u = C\mathcal{E} - Q$$

$$\frac{du}{dQ} = -1$$

$$du = -dQ$$

$$\int \frac{-du}{u} = \frac{1}{RC}\int dt$$

$$-\ln|u| + K = \frac{1}{RC}(t + k)$$

$$-\ln|C\varepsilon - Q| + K = \frac{t}{RC} + k$$

$$-\ln|C\varepsilon - Q| = \frac{t}{RC} + k$$

$$e^{\ln|C\varepsilon - Q|} = e^{-\frac{t}{RC} + k}$$

$$C\varepsilon - Q = e^{-t/RC + k}$$

$$C\varepsilon - Q = e^{-t/RC} \cdot e^{k}$$

$$C\varepsilon - Q = Ke^{-t/RC}$$

$$-Q = Ke^{-t/RC} - C\varepsilon$$

$$Q = Ke^{-t/RC} + C\varepsilon$$

Find k by plugging in initial conditions

$$Q(0) = 0$$

$$0 = Ke^{0} + C\varepsilon$$

$$0 = k + C\varepsilon$$

$$k = -C\varepsilon$$

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$$Q(t) = C\mathcal{E}\left[1 - e^{-t/RC}\right]$$

to solve for voltage, use $V = \frac{Q}{C}$

$$V(t) = \frac{Q(t)}{C}$$

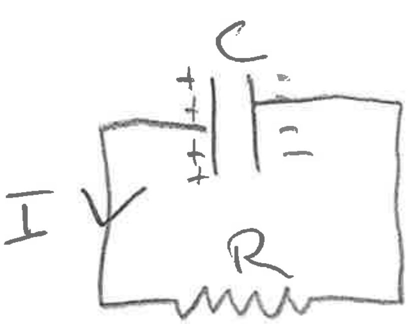
$$V(t) = \frac{C\mathcal{E}\left[1 - e^{-t/RC}\right]}{C}$$

$$V(t) = \mathcal{E}\left[1 - e^{-t/RC}\right]$$

Define scale time $\tau = RC$ (RC has units seconds).

$$\boxed{V(t) = \mathcal{E}\left[1 - e^{-t/\tau}\right] \text{ (charging)}}$$

Discharging case



Loop Rule

$$V_C - IR = 0$$

- current in circuit is negative time-derivative of charge on capacitor. 5

$$I = -\frac{dQ}{dt}$$

$$V_c - -\frac{dQ}{dt} R = 0$$

$$\frac{Q}{C} + \frac{dQ}{dt} R = 0$$

Put Q terms on one side, t terms on other.

$$\frac{Q}{C} = -\frac{dQ}{dt} R$$

$$\frac{Q}{RC} = -\frac{dQ}{dt}$$

$$\int -\frac{dQ}{Q} = \int \frac{dt}{RC}$$

$$-\ln|Q| = \frac{t}{RC} + k$$

$$\ln|Q| = -\frac{t}{RC} + k$$

$$Q = e^{-t/RC + k}$$

$Q = ke^{-t/RC}$

Plug in initial conditions

$Q(0) = Q_{max}$

$Q_{max} = ke^{0}$

$k = Q_{max}$

$Q = Q_{max} e^{-t/RC}$

$Q = CV$

$CV(t) = Q_{max} e^{-t/RC}$

$V(t) = \frac{Q_{max}}{C} e^{-t/RC}$

$\frac{Q_{max}}{C}$ is the voltage capacitor was charged to.

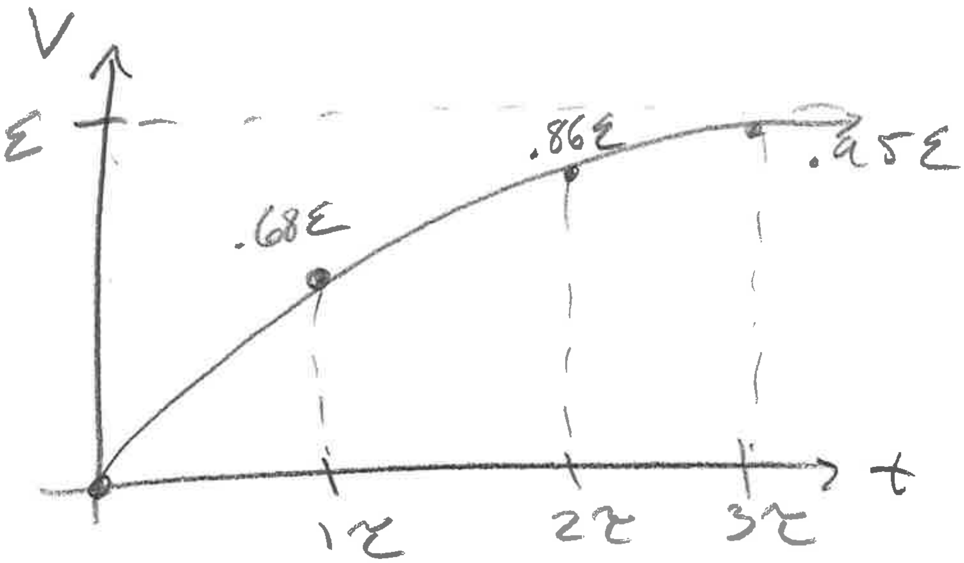
$V(t) = \mathcal{E} e^{-t/RC}$

$RC = \tau$

$\boxed{V(t) = \mathcal{E} e^{-t/\tau} \text{ (discharging)}}$

Charging

$$V = \mathcal{E}\left[1 - e^{-t/\tau}\right]$$



$t \to \infty$

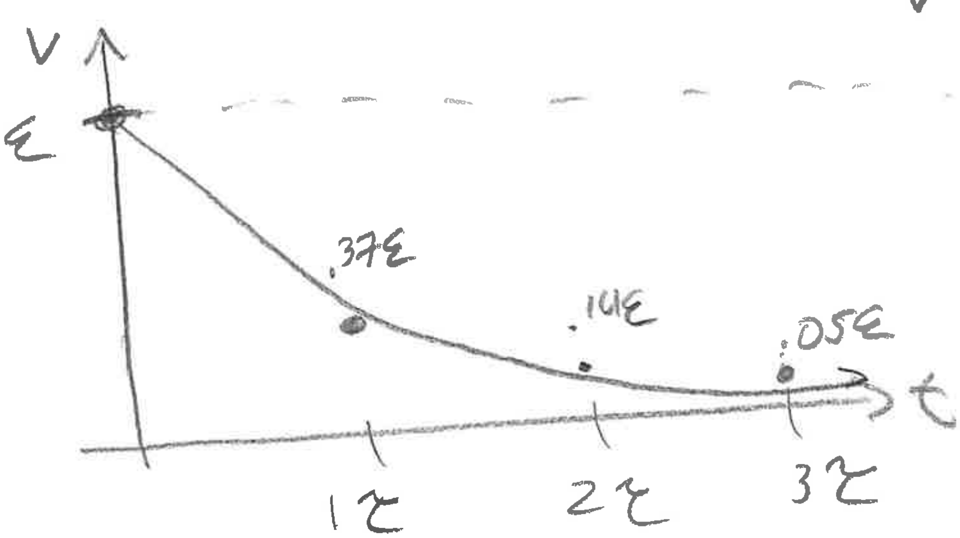
$V \to \mathcal{E}$

Discharging

$$V = \mathcal{E}\, e^{-t/RC}$$

$t \to \infty$

$V \to 0$



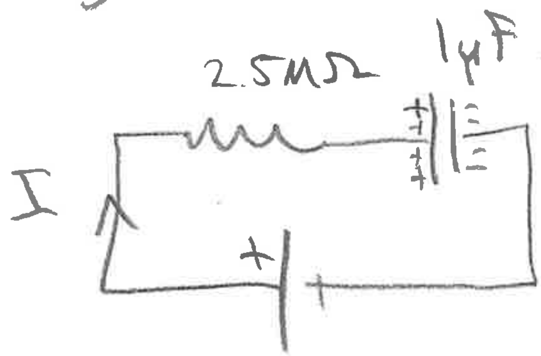
Shape of curves dictated by time constant $\tau = RC$!

RC Circuit Example

1. A 12V battery is connected to 2.5 MΩ resistor and a 1μF capacitor.

   a) Find $\tau = RC$

   b) Calculate the time to charge the capacitor 75% its full charge.



a) $\tau = RC$

$\tau = (2.5 \times 10^{6})(1 \times 10^{-6})$

$\boxed{\tau = 2.5\,s}$

b) $V(t) = 0.75\,\mathcal{E}$

$t = ?$

$$V(t) = \mathcal{E}\left[1 - e^{-t/RC}\right]$$

$$\frac{V(t)}{\mathcal{E}} = 1 - e^{-t/RC}$$

$$\frac{V(t)}{\mathcal{E}} - 1 = -e^{-t/\tau}$$

$$\ln\left|e^{-t/\tau}\right| = \ln\left|1 - \frac{V(t)}{\mathcal{E}}\right|$$

$$-\frac{t}{\tau} = \ln\left|1 - \frac{V(t)}{\mathcal{E}}\right|$$

$$t = -\tau \ln\left|1 - \frac{V(t)}{\mathcal{E}}\right|$$

$V(t) = 0.75\,\mathcal{E}$

$\tau = 2.5s$

$$t = -2.5 \ln\left|1 - .75\right|$$

$\boxed{t = 3.47s}$ $\quad 1\tau < t < 2\tau$ as expected.