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Magnetic braking: Improved theory

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An alternative analysis is presented for the magnetic braking experiment of Wiederick *et al.*, taking into account the fringing streamlines of eddy currents for a rectangular "footprint" of the magnetic field.

In a recent article,[1] Wiederick *et al.* described an instructive experimental project concerning the magnetic braking of a rotating conducting disk. Their theoretical analysis made two simplifications: (1) an infinite sheet, translating linearly rather than rotating, and (2) uniform eddy-current density within the rectangular "footprint" of the magnetic field. The purpose of the present article is to replace the second simplification by a more realistic theory that allows for the fact that the eddy currents tend to concentrate toward the ends of the footprint and are not orthogonal to the velocity of the conducting sheet. This analysis removes the need for a separate evaluation of the external resistance R, which the earlier authors did by conformal mapping. The geometry and notation used here follow Ref. 1.

In the lab frame (the frame of the magnet), the current density induced in the disk translating at constant velocity $\mathbf{v}$ is given by

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{1}$$

where $\mathbf{B} = B_0 \hat{\mathbf{k}}$ within the magnet footprint of length l by width w and zero outside, and $\mathbf{E}$ is the electrostatic field of Coulomb charge induced within the conducting sheet along the edges of the footprint parallel to the motion of the sheet. (The magnet gap, in which the conducting sheet moves, is very thin compared to l and w, so that the magnetic field falls abruptly to zero at the edges of the footprint or "shadow" of the pole pieces.) As discussed in Ref. 1, the $\mathbf{v} \times \mathbf{B}$ term is an electromotive field driving the eddy currents $\mathbf{J}$.

Figure 1, computed from the calculation described below, shows the streamlines of $\mathbf{J}$ that result from the uniform, rectangular $\mathbf{v} \times \mathbf{B}$ driving field. As is to be expected for finite rectangular geometry, the current "fringes" near the ends of the footprint, increasing in magnitude there and no longer lying parallel to $\mathbf{v} \times \mathbf{B}$. Often fringe phenomena of this sort are difficult to calculate but, in this case, an analytic solution can be obtained.

The current density $\mathbf{J}$ of the present problem bears a close formal analogy to the $\mathbf{B}$ field of a uniformly magnetized sample of magnetic material.[2] In that case,

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \tag{2}$$

where the magnetization $\mathbf{M}$ is the causal agent, analogous to $\mathbf{v} \times \mathbf{B}$ in Eq. (1). The simplest strategy in the magnetic analog is to calculate the $\mathbf{H}$ field, inside and outside the material, by integrating the Coulomb field (or scalar potential) of a magnetic pole density $\mathbf{M} \cdot \hat{\mathbf{n}}$ over the surface of the uniformly magnetized sample ($\hat{\mathbf{n}}$ = unit normal), and then summing $\mathbf{H}$ and $\mathbf{M}$ according to Eq. (2). An appropriate figure is shown in many electromagnetism texts.[3]

In the present case, the Coulomb sources of $\mathbf{E}$ are a surface charge (charge/area) of magnitude $\pm \epsilon_0 v B_0$ on planes parallel to the y–z axes at $x = \pm w/2$, within the conducting sheet. From this, we will calculate the fringed $\mathbf{E}$ field in the x–y plane (our Fig. 1). First we note a subtlety with respect to the z dimension. Since the geometry is very finite in that dimension, one might expect fringing effects there also. However, in the steady state, the currents within the sheet (of thickness δ) can have no component in the z direction. In the initial transient as the (quasiconstant) velocity of the sheet is established, electrostatic charge accumulates on the surfaces of the sheet in such a way as to

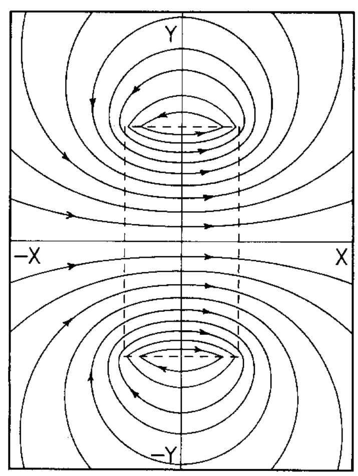

Fig. 1. Eddy-current streamlines in a conducting sheet, moving in the y direction. The uniform magnetic field is in the z direction, normal to the figure, within the dashed rectangle of x width w and y length l.

cancel the "fringe" z components of the interior electric field. The net result is that the interior $\mathbf{E}$ field is that of capacitorlike surface charges $\pm\epsilon_0 vB_0$ at $x=\pm w/2$ that extend indefinitely in the z direction, and from $-l/2$ to $+l/2$ in the y direction.

To calculate the $\mathbf{E}$ field within the conducting sheet, we integrate the $1/r$ field of line charges extending indefinitely in the z direction and of charge/length equal to $\pm\epsilon_0 vB_0 d\eta$, from $\eta=-l/2$ to $+l/2$. For notational ease, we introduce the variables $a=w/2$ and $b=l/2$. Then

$$E_x(x,y)=\frac{vB_0}{2\pi}\int_{-b}^{+b}\left(\frac{(x-a)}{(y-\eta)^2+(x-a)^2}-\frac{(x+a)}{(y-\eta)^2+(x+a)^2}\right)d\eta$$
$$=-\frac{vB_0}{2\pi}\left(\tan^{-1}\frac{b+y}{a-x}+\tan^{-1}\frac{b-y}{a-x}+\tan^{-1}\frac{b+y}{a+x}+\tan^{-1}\frac{b-y}{a+x}\right). \quad (3)$$

Similarly,

$$E_y(x,y)=\frac{vB_0}{4\pi}\ln\frac{[(a-x)^2+(b+y)^2][(a+x)^2+(b-y)^2]}{[(a+x)^2+(b+y)^2][(a-x)^2+(b-y)^2]}. \quad (4)$$

These formulas apply both inside and outside the magnet footprint. The current density is

$$J_x=\begin{cases}\sigma(E_x+vB_0) & \text{(inside)},\\ \sigma E_x & \text{(outside)},\end{cases} \quad (5)$$

$$J_y=\sigma E_y \quad \text{(everywhere)}. \quad (6)$$

The magnetic braking force is then

$$\mathbf{F}=\int \mathbf{J}\times\mathbf{B}_0\,d\tau=-\sigma\delta B_0\int_{-a}^{a}\int_{-b}^{b}(E_x+vB_0)dx\,dy\hat{\mathbf{j}}$$
$$=-\alpha(\sigma\delta B_0^2 lw)v\hat{\mathbf{j}}, \quad (7)$$

as given by Eq. (7) of Ref. 1, but now with the coefficient α given by

$$\alpha=1-\frac{1}{2\pi}\left[4\tan^{-1}A+A\ln\left(1+\frac{1}{A^2}\right)-\frac{1}{A}\ln(1+A^2)\right], \quad (8)$$

where $A=l/w$ is the aspect ratio of the magnet footprint. In Ref. 1, the coefficient α was expressed as $(1+R/r)^{-1}$, where $r=w/\sigma\delta l$ is the "internal resistance" of the rectangular conductor within the magnetic field, and R is the resistance of the external circuit for the conducting sheet with a rectangular hole cut out of it. We note that in the present analysis there is no well-defined "terminal voltage" of the form of Eq. (2) of Ref. 1. If one wishes to compare the two analyses, we can write $R=(1-\alpha)/\alpha A\sigma\delta$.

For $A=1$, Eq. (8) gives $\alpha=\frac{1}{2}$. It should be noted that $\alpha(A)+\alpha(1/A)=1$. For $l\gg w$, α goes to zero as $\alpha\to(1.5+\ln A)/\pi A$, and thus for $l\ll w$, $\alpha\to 1$. The $l\gg w$ limit gives $F\sim lw/A=w^2$, showing that the eddy-current braking is largely an end effect, which may be understood physically by noting that partial currents driven by $\mathbf{v}\times\mathbf{B}$ near the middle of a long footprint ($l\gg w$) return within the footprint and thus do not contribute to the net drag force. A reasonably good empirical approximation to Eq. (8) is $\alpha=1/(2\sqrt{A})$ over the range $1\leqslant(A=l/w)\leqslant 2$.

We may compare this analysis with the experimental measurement reported in Ref. 1. For $A=57/29=1.9655$, Eq. (8) gives $\alpha=0.3557$. The $\alpha=(1+R/r)^{-1}$ of Ref. 1 is computed from $r=1/\sigma\delta A$ and $R=0.87/\sigma\delta$ (from conformal mapping) to give $\alpha=0.369$. The difference of 3.7% is in the correct sense to remove most of the 3.1% discrepancy between their experimental and theoretical results. The remaining discrepancy is probably due to the experimental truncation of the "infinite" sheet.

Our Fig. 1 was produced using a computer line-tracing routine to plot streamlines of $\mathbf{J}$ from Eqs. (5) and (6). The spacing of the streamlines was chosen by integrating $J_x(x=0;y)$ from 0 to η ($\eta\leqslant l/2$), and finding values of η such that the total current is divided into ten equal parts. The discontinuity in J_x at the $y=\pm l/2$ boundaries arises from the assumed sharp discontinuity in B_0 at the edge of the magnet footprint. Along the $x=\pm w/2$ boundaries, the effect on J_x of the B_0 discontinuity is canceled by the net charge induced in the moving sheet.

[1] H. D. Wiederick, N. Gauthier, D. A. Campbell, and P. Rochon, Am. J. Phys. **55**, 500 (1987).

[2] See, for instance, W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1962), 2nd ed., pp. 140–144; P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves* (Freeman, San Francisco, 1970), 2nd ed., pp. 113–115, 402–405.

[3] For instance, from Ref. 2, see Panofsky and Phillips, Fig. 8-1, p. 142; and Lorrain and Corson, Fig. 9-11, p. 403.