One can not insist sufficiently on the fact that the special theory of relativity today rests upon innumerable experimental verifications, for we can regularly obtain particles of velocities approaching that of light in vacuum, particles in regard to which it is necessary to take account of corrections introduced by the special theory of relativity.

L. DE BROGLIE (1949)

From an essay in *Albert Einstein: Philosopher-Scientist*, P. A. Schilpp, ed., a collection of essays published in honor of Einstein's 70th birthday, Harper Torchbooks, New York, 1959.

6

Relativistic dynamics—collisions and conservation laws

FACED WITH the results of special relativity, we should in principle rewrite all our mechanics accordingly. But we know that this is not necessary. The Newtonian scheme, although it is strictly correct only in the limit of vanishingly small velocities, works beautifully in an enormous variety of situations. This, as we have seen, is because the greatest velocities that we encounter in the dynamics of ordinary macroscopic objects are still minute compared to the velocity of light ($v < 10^{-5}c$). There is, however, one area in which the use of special relativity is clearly called for—in problems involving velocities that are *not* negligible compared with the velocity of light. And this means, primarily, the world of atomic and nuclear particles. It is with such problems, therefore, that this chapter will be largely concerned.

We shall not attempt in this chapter to consider in detail the motions of particles under the action of specified forces. Our goal will be a more modest one. What we shall do is to show the kinds of calculations one can do with the help of just two principles: (1) conservation of linear momentum, and (2) conservation of energy. No. (1) will, as in the familiar Newtonian problems, be applicable to each of the three separate components of linear momentum or to the total momentum treated as a single vector. No. (2) is used with the understanding that the total energy in all forms, including mass, is the conserved quantity. We shall take it as basic that these two principles apply to any self-contained system, and we shall concentrate on situations in which an interaction is over and done with in some limited span of time. In other words, we shall fix our attention on collisions or analogous processes, and our only concern will be to relate "before" and "after."

after collision

Before being able to apply these conservation principles, however, we must consider how to formulate and justify them in relativistic terms. In Chapter 1 we developed expressions for the mass, momentum, and total energy of a single particle of rest mass m_0 moving at speed v relative to the laboratory:

$$
\begin{aligned}
m(v) &= \gamma m_0 \\
\mathbf{p} &= \gamma m_0 \mathbf{v} \qquad\qquad (6\text{–}1)\\
E &= \gamma m_0 c^2
\end{aligned}
$$

with $\gamma(v) = (1 - v^2/c^2)^{-1/2}$. The derivation of these results made explicit use of the relation between energy and momentum for photons ($p = E/c$). Furthermore, if you consider the arguments in detail, you will see that, in fact, we *assumed* that conservation of momentum and energy held good—and then inferred the appropriate formulas for momentum and energy required by this assumption. Thus in considering the pressure of light experiment, we could not have inferred anything about the momentum of photons without assuming that this momentum was fully transferred to the illuminated surface. In the Einstein box calculation, also, the conservation of momentum is explicitly assumed. In discussing the ultimate speed experiment, we took it for granted that a calorimetric measurement would, through energy conservation, give us an exact knowledge of the kinetic energy brought in by electrons traveling at speeds close to the speed of light.

At this stage, therefore, we are not *discovering* these grand conservation principles of dynamics; we are, instead, *asserting* them, on the grounds that the use of such principles has already been amply justified in classical dynamics. And then, in going from Newtonian to Einsteinian dynamics, we are simply extending the range of problems that can be handled according to a single set of rules. In the process we arrive at new prescriptions for calculating such quantities as momentum and kinetic energy in terms of the velocity and an inertial parameter (the mass).

no proof, just assertion

The transition to relativistic dynamics is not, however, an arbitrary one. Given the relativistic kinematics, and a knowledge of the Newtonian laws, one is led quite naturally to the relativistic formulation. We were not in a position to do this in Chapter 1, and our arguments there were, as we pointed out, more suggestive than convincing. But at this point we can offer a much cleaner approach, based upon readily visualizable situations which we shall now discuss. The course of the argument illustrates, once again, the very intimate connection between the particular formulation of kinematics and the dynamics appropriate to it.

TWO VIEWS OF AN ELASTIC COLLISION

simplest elastic collision

We are going to consider a very simple type of collision process—a perfectly elastic collision between two identical particles. It will be a collision in which the whole motion takes place in one plane, and we shall analyze it in terms of momentum conservation.

By way of background, consider for a moment the Newtonian version of this process. Bodies A and B, with initial velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, collide with one another and afterwards have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. In any individual collision of this type, it is always possible to find a set of four scalar multipliers (α) that permit one to write an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 = \alpha_3\mathbf{v}_1 + \alpha_4\mathbf{v}_2$$

This as it stands is a quite uninteresting statement. But experiments for all sorts of values of $\mathbf{u}_1$ and $\mathbf{u}_2$ reveal the remarkable result that in every such collision, for two given objects, we can obtain a vector identity by putting $\alpha_1 = \alpha_3 = m_A$ (a scalar property of body A) and $\alpha_2 = \alpha_4 = m_B$ (the corresponding scalar property of body B). In other words, the purely kinematic observations on a collision process lead us to introduce the parameters that we call the *inertial masses* of the two bodies, and permit us to write the familiar equation for conservation of linear momentum.

When we introduce the relativistic kinematics, the relationship between initial and final velocities for two colliding objects is no longer expressible in quite such a simple form. Nevertheless, we keep as close to it as we can, and we do this by asking what is implied by the kinematics of such a collision if we assert conservation of linear momentum in the following extended sense.

In the elastic collision of two bodies, A and B, as described from the standpoint of a particular frame of reference, the initial and final velocities are related by the equation

$$m_A(u_1)\mathbf{u}_1 + m_B(u_2)\mathbf{u}_2 = m_A(v_1)\mathbf{v}_1 + m_B(v_2)\mathbf{v}_2 \tag{6–2}$$

where m_A is a scalar inertial property of A depending only on its speed and m_B is the corresponding scalar property of B. We know that in Newtonian mechanics this equation is satisfied by values of m_A and m_B that are quite independent of speed. Let us now see how the relativistic kinematics implies the dependence of m on v as given in Eq. (6–1).

We imagine two experimenters, one in the inertial frame S and the other in S'. They use identical types of instruments for measuring times and distances, and they agree to produce a completely symmetrical collision between two identical particles.[1] The experimenter in S will project one particle (A) along his y axis with a speed u_0 (as measured in S), and the experimenter in S' will project the other particle (B) along his y' axis with a speed $-u_0$ (as measured in S'). *The speed u_0 is small, but S and S' have a very large relative velocity v along x.* The experimenters are so skillful that the particles collide when their centers lie along the y axis. The collision as observed in S and S' thus takes the forms shown in Fig. 6–1. The y (or y') component of velocity of each

[1]A gedanken experiment of this type was first introduced by G. N. Lewis and R. C. Tolman, *Phil. Mag.*, **18,** 510 (1909).

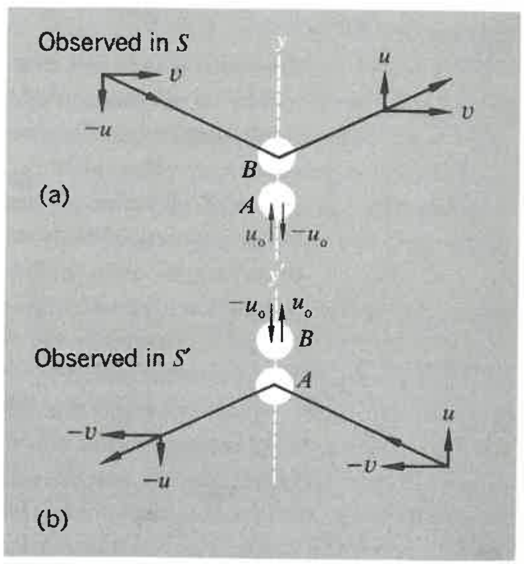

Fig. 6–1 Elastic collision between two identical objects, observed from two reference frames related by a velocity v along x.

particle is exactly reversed, and its velocity along x (or x') is unchanged. This corresponds to complete reversibility in time for the collision as a whole.

In analyzing this process, we note the following points:

1. As observed in S, the y component of velocity of A is initially u_0, and afterward $-u_0$. The y component of velocity of B is initially $-u$, and afterward u. The relation between u_0 and u is given by the transformation equation [Eq. (5–3)] for velocity components transverse to the direction of relative motion of two reference frames:

$$u_y = \frac{u_y'/\gamma}{1 + vu_x'/c^2}$$

Since in the frame S' the velocity component u_x' of B is zero, this reduces to

$$u = u_0/\gamma = u_0(1 - v^2/c^2)^{1/2} \tag{6–3}$$

2. As observed in S', the roles of A and B are interchanged and the sign of v is reversed. The complete symmetry can be clearly recognized if Fig. 6–1(b) is rotated through 180° in its own plane; it then matches Fig. 6–1(a) in every respect.

3. As observed in either reference frame, the *speed* of each particle remains unchanged by the collision, and is either u_0 or $(u^2 + v^2)^{1/2}$. Because of this, and the identity of the particles, we are concerned with only two possible values of m—$m(u_0)$ and $m(V)$, where $V = (u^2 + v^2)^{1/2}$.

The conservation of linear momentum in the y direction as observed in S is then described by the following statement:

$$p_y = m(u_0)u_0 - m(V)u = -m(u_0)u_0 + m(V)u$$

Therefore,

$$\frac{m(V)}{m(u_0)} = \frac{u_0}{u} \tag{6–4}$$

Now we have postulated that u_0 is small—as small as we choose to imagine. Hence the inertial quantity $m(u_0)$ can be taken to be just the Newtonian inertial mass m_0. Also, given that $u_0 \ll v$, it follows (a fortiori) from Eq. (6–3) that $u \ll v$, and hence that $V \approx v$. Thus, by imagining a limiting collision of this type, with $u_0 \to 0$, we conclude from Eq. (6–4) that

$$m(v) = \gamma m_0 = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \tag{6–5}$$

and hence that $\mathbf{p} = \gamma m_0 \mathbf{v}$ is an appropriate definition of the linear momentum for a particle of rest mass m_0 traveling at velocity $\mathbf{v}$.

The above discussion is limited strictly to the question of momentum. Is the definition of mass that emerges from this analysis—i.e., as given by Eq. (6–5)—also applicable to calculations involving the *energy* of the system? We have of course already argued in Chapter 1 that this is indeed so, but an analysis based upon another hypothetical collision process will perhaps lend further conviction to the result.

TWO VIEWS OF AN INELASTIC COLLISION

Again we shall consider the impact of two identical particles, but this time we shall suppose that the collision is completely *in*elastic. There will be a frame S' in which the particles approach each other along a straight line with equal and opposite velocities of magnitude u [Fig. 6–2(a)]. There will then exist another frame S, relative to which S' has velocity u, in which one of the particles is initially stationary [Fig. 6–2(b)].

As observed in frame S', the collision results in the formation of a stationary composite particle. Hence in frame S this composite particle must be observed to have the velocity u. In this same frame S the initially moving particle has a velocity U related to u through the velocity addition formula, Eq. (5–2):

$$u_x = \frac{u_x' + v}{1 + vu_x'/c^2}$$

in which we put $u_x = U$, $u_x' = v = u$. Therefore,

$$U = \frac{2u}{1 + u^2/c^2} \qquad (6\text{–}6)$$

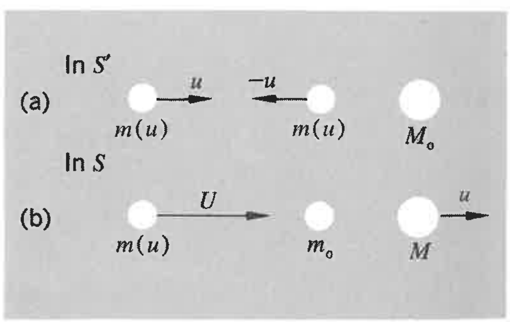

Fig. 6–2 Completely inelastic collision between two similar objects, observed (a) in the zero-momentum frame, (b) in a frame in which one of the objects is initially stationary.

Now let us write the statements of conservation of momentum and conservation of *mass*, from the standpoint of frame S. (As in the previous section, we suppose that the mass is some function of the speed):

Momentum:

$$m(U)U = Mu \qquad (6\text{–}7)$$

Mass:

$$m(U) + m_0 = M \qquad (6\text{–}8)$$

Eliminating M between these two equations we find

$$\frac{m(U)}{m_0} = \frac{u}{U - u} \qquad (6\text{–}9)$$

Now Eq. (6–6) gives the connection between U and u; thus we can find the ratio $m(U)/m_0$ as an explicit function of U. From Eq. (6–6) we have

$$u^2 - 2(c^2/U)\,u + c^2 = 0$$ ✓ just re-arr. 6.6 to left and norm. highest pow term (like quad)

Therefore,

$$u = \frac{c^2}{U} \pm \left[\left(\frac{c^2}{U}\right)^2 - c^2\right]^{1/2}$$
$$= \frac{c^2}{U}[1 \pm (1 - U^2/c^2)^{1/2}]$$

Since we must have $u \to U/2$ for $U \ll c$, we know that the negative sign should be chosen. (Appeal to the fact that the radical is approximately equal to $1 - U^2/2c^2$ for $U \ll c$.) Thus we have

$$u = \frac{c^2}{U}[1 - (1 - U^2/c^2)^{1/2}] \qquad (6\text{–}10a)$$

Therefore,

$$U - u = \frac{c^2}{U}[U^2/c^2 - 1 + (1 - U^2/c^2)^{1/2}]$$

i.e.,

$$U - u = \frac{c^2}{U}(1 - U^2/c^2)^{1/2}[1 - (1 - U^2/c^2)^{1/2}] \qquad (6\text{–}10b)$$

Substituting from equations (6–10) into (6–9), we have

$$\frac{m(U)}{m_0} = \frac{1}{(1 - U^2/c^2)^{1/2}} = \gamma(U) \qquad (6\text{–}11)$$ ✓

which thus reproduces the form of the mass formula of Eq. (6–5).

This calculation involves more algebraic manipulation than the one in the previous section, but it is more satisfactory in several ways:

1. The collision considered is an extremely simple one, the motion being entirely along a single straight line.

2. The calculation is exact. One of the particles is, by definition, completely stationary in frame S before the collision (not just approximately so as it was in the previous case).

3. The explicit use of the mass-conservation equation leads naturally to the equivalence of mass and energy. We have, in essence, already developed the connection in these terms in Chapter 1, so the chief purpose of restating it here is to emphasize once again the intimate connection between the kinematics and the dynamics. From Eq. (6–11) we have

$$\begin{aligned} m(U) &= m_0(1 - U^2/c^2)^{-1/2} \\ &= m_0 + \tfrac{1}{2}m_0U^2/c^2 + \cdots \end{aligned}$$

Therefore,

$$m(U)c^2 = m_0c^2 + \tfrac{1}{2}m_0U^2 + \cdots \tag{6–12}$$

Noting that the second term on the right of Eq. (6–12) corresponds exactly to the classical kinetic energy of a particle of mass m_0 and speed U, we come to the familiar statement that the *total* energy of a particle of rest mass m_0 and speed U is given by

$$E = \frac{m_0c^2}{(1 - U^2/c^2)^{1/2}} = m(U)c^2 \tag{6–13}$$

with $E_0 = m_0c^2$ defining the rest energy of the particle.

4. By considering the collision further, we can demonstrate that the *consistent* use of a mass/velocity relation as given by Eq. (6–11) involves no contradictions. In Eq. (6–11) we have a statement of the mass of that colliding particle which, as observed in frame S, has speed U. Let us express this in terms of u, using Eq. (6–6); we have

$$\begin{aligned} 1 - U^2/c^2 &= 1 - \frac{4(u^2/c^2)}{(1 + u^2/c^2)^2} \\ &= \frac{(1 - u^2/c^2)^2}{(1 + u^2/c^2)^2} \end{aligned}$$

Therefore,

$$m(U) = \frac{(1 + u^2/c^2)}{(1 - u^2/c^2)}\, m_0$$

Substituting this value in Eq. (6–8), we find that the mass of the composite particle, as measured in frame S, is given by

$$M = \frac{2m_0}{1 - u^2/c^2} \tag{6–14}$$

But in frame S this composite particle has speed u. On the basis of Eq. (6–11) we would infer that its rest mass should be M_0, where

$$M_0 = M(1 - u^2/c^2)^{1/2}$$

Using Eq. (6–14), this would give us

$$M_0 = \frac{2m_0}{(1 - u^2/c^2)^{1/2}} \tag{6–15}$$

But now, consider the collision as described in the frame S'. Here the composite particle is indeed at rest, having been formed from the collision of two particles, each of rest mass m_0 and speed u. We are assuming that all the mechanical energy brought in by the colliding particles is retained within the composite particle. Thus we do not (and must not) assume that M_0 is equal to $2m_0$. Using the statement of conservation of mass as applied in frame S', we have

$$M_0 = 2m(u) \tag{6–16}$$

which is identical with Eq. (6–15) if we accept the velocity dependence of mass as given in Eq. (6–11) and thus put $m(u) = \gamma(u)m_0$.

FURTHER REMARKS ON THE CONSERVATION LAWS

It should be very clear from all the preceding discussion that the momentum and energy conservation laws are not sacred; there is nothing, however, in our experience so far that has required their abandonment. It has been a pretty near thing at times—as, for instance, when the existence of the neutrino, a hitherto unobserved particle, was postulated by W. Pauli in 1930 to avoid giving up conservation of energy in beta decay. It took over 20 years before the neutrino was detected—but it was,[1] and our confidence in the conservation laws was still further strengthened thereby.

Clearly the conservation laws are not going to tell us the

[1] F. Reines, et al., *Phys. Rev.*, **92**, 830 (1953).

whole story by a long way. In the collision of two atomic particles, for example, we shall need detailed information about interatomic forces before being able to answer many of the important questions: Will the particles stick together? Or will they undergo an elastic collision? Or a partially inelastic collision? What will be the probability of scattering in a particular direction? What is the total effective target area that they present to each other? Will radiation be emitted after impact, and if so, of what kind? To provide the answers to these questions over a wide range of conditions is likely to require many man-years of research. Nevertheless, the basic conservation laws are the essential foundation for all else, and their generality makes them a powerful tool. What we shall do, then, is to hitch ourselves to the following statements (for an isolated system) and see where they lead us:

$$E_{\text{total}} = c^2 m_{\text{total}} = \text{constant} \tag{6-17}$$

$$\mathbf{p}_{\text{total}} = \text{constant} \tag{6-18}$$

We shall also be making use of the dynamic relations, already introduced in Chapter 1, that are derivable from equations (6–1):

$$E^2 = (cp)^2 + E_0{}^2 \qquad (\text{with } E = mc^2,\ E_0 = m_0c^2) \tag{6-19}$$

$$v = \frac{p}{m} = \frac{c^2 p}{E}$$

or

$$\beta = \frac{v}{c} = \frac{p}{mc} = \frac{cp}{E} \tag{6-20}$$

Forces and accelerations will not enter this part of our discussion at all.

We shall begin with some collision problems involving photons, because their lack of any rest mass makes the equations less complicated.

ABSORPTION AND EMISSION OF PHOTONS

Absorption

Suppose that a stationary particle (e.g., an atom or nucleus) of mass (rest mass) M_0 is struck by a photon (quantum) of energy Q which is completely absorbed. The combined system will have mass M' and will recoil with a velocity v. Then we have