T transforms 4-vector comp. b/w frames

The Lorentz transformations can then be regarded as a prescription for transforming the various components of a 4-vector from one set of axes to another—often described as the "mapping" of the vector onto various coordinate systems in the four-dimensional world. This way of representing the relativistic scheme of things is attractive in a formal sense, and can also be very useful if one has once learned to exploit it—which primarily means developing the appropriate fluency in matrix algebra. We shall not go any further with it here, however, since it is not essential and really adds nothing to the basic physics of relativity.

FORCE IN RELATIVISTIC MECHANICS

not necess. co-dir.

In contrast to the Newtonian conception, it is easy to show that in relativity the quantity force, in general, is not codirectional with the acceleration it produces . . . It is also easy to show that these force components have no simple transformation properties . . . All these modifications, important as they are from the mathematical point of view, do not radically affect the conception of force. Yet an important point should be noted: on grounds of the rejection of an absolute simultaneity of two distant events, special relativity comes to the conclusion that action at a distance has to be excluded as a legitimate physical notion.

no action at a distance

Max Jammer, *Concepts of Force* (1957)

In all of our discussion of relativistic dynamics so far, we have placed an almost exclusive emphasis on the use of the energy and momentum conservation laws for an isolated system of particles. We have tried to give some feeling for the variety of problems that can be discussed in these terms. But when all is said and done, this approach is not always the most convenient or useful. Many, many problems in dynamics can best be treated (and perhaps can only be efficiently treated) in terms of the motion of particles under the action of a given set of forces. Take the Rutherford scattering of α particles, for example. Granted that the force becomes vanishingly small at large separations between two particles (and it is this assumption that underlies all the calculations we have done in Chapter 6 and in this chapter), one can make perfectly correct statements about the relation between final directions and velocities of the colliding nuclei. But this does not tell us the probability that an α particle will in fact be deflected through a certain angle. Only when we put in the explicit law of force do we find answers to such questions as these. And the discovery and specification of laws of force is a central concern of physics. It is certainly important, therefore, to know how to transform forces and equations of motion so as to give a description of them from the point of view of different inertial frames. Since in special relativity the acceleration is not an invariant, we know that we cannot enjoy the simplicity of Newtonian mechanics, but we can certainly arrive at some useful and meaningful statements.

The starting point, which we indeed made use of in the initial stages of our approach to relativity (see Chapter 1) is Newton's law in the form

def. F

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) \qquad \text{where } m = m_0(1 - v^2/c^2)^{-1/2} \tag{7–16}$$

We take this as a definition of **F**. It is a natural extension (and the simplest extension) of the nonrelativistic result. It is not a statement that can be independently proved. On the other hand, if the analytical form of **F** is given, in terms of coordinates, velocities, etc., we must certainly demand that the left and right sides of Eq. (7–16) transform in the same way under Lorentz transformations. Assuming that this necessary condition has been met, the transformations of the components of $d\mathbf{p}/dt$ tell us how force components transform in special relativity.

req. of F

$\frac{dp_i}{dt} \to F_i$

Our approach to the problem will be as follows: At any instant a particle has a well-defined velocity **v** as measured in a laboratory frame of reference. We can picture the particle as being instantaneously in a rest frame that has this velocity **v** with respect to the laboratory. We shall imagine that, as measured in the rest frame, a force F_{0x} is applied parallel to **v**, causing an acceleration a_{0x}. The mass as measured in this frame is just the rest mass m_0. Hence we have

$$F_{0x} = m_0 a_{0x} \tag{7–17}$$

Now in the laboratory frame we have a momentum given by

$$p_x = \gamma m_0 v = \frac{m_0 v}{(1 - v^2/c^2)^{1/2}}$$

and hence we judge the force to be F_x, where

$$F_x = \frac{dp_x}{dt} = \frac{m_0}{(1 - v^2/c^2)^{1/2}}\frac{dv}{dt} + m_0 v \frac{d}{dt}[(1 - v^2/c^2)^{-1/2}]$$

If we put $dv/dt = a_x$ (the observed acceleration in the laboratory) we have

$$F_x = \frac{m_0 a_x}{(1 - v^2/c^2)^{1/2}} + \frac{m_0(v^2/c^2)a_x}{(1 - v^2/c^2)^{3/2}}$$

which, when we collect terms, simplifies to

$$F_x = \gamma^3 m_0 a_x \tag{7–18}$$

There is, however, a very simple connection between a_x and a_{0x}:

$$a_x = \frac{1}{\gamma^3} a_{0x} \tag{7–19}$$

This is the particular case of the transformation of accelerations along x for $u_{0x} = 0$ [cf. Eq. (5–24)]. Thus Eq. (7–18) can be written

$$F_x = \gamma^3 m_0 \frac{a_{0x}}{\gamma^3} = m_0 a_{0x}$$

i.e.,

$$F_x = F_{0x} \tag{7–20}$$

This is a striking result. Despite the change of the measures of mass and acceleration in the two frames, the measure of the x component of force remains the same.

When we make a similar calculation for the transverse force, we find that this invariance does not hold. In the instantaneous rest frame we have

$$F_{0y} = m_0 a_{0y} \tag{7–21}$$

In the laboratory frame, the force F_y applied perpendicular to the momentum vector $m\mathbf{v}$ will, during some very brief interval of time, leave the magnitude of the velocity unaltered; it merely changes the direction of $\mathbf{v}$ slightly by introducing a small transverse component. Thus to a good approximation (which becomes perfect in the limit $\Delta t \to 0$) the mass remains unchanged at γm_0, and the transverse impulse can be written

$$F_y \, \Delta t = \gamma m_0 \, \Delta v_y$$

Hence we have, in this case,

$$F_y = \gamma m_0 a_y \tag{7–22}$$

Again there is a simple relation between the accelerations in the two frames, given that one of them is measured in the rest frame:

$$a_y = \frac{1}{\gamma^2} a_{0y} \tag{7–23}$$

this being a special case of Eq. (5–26). We therefore have

$$F_y = \gamma m_0 \frac{a_{0y}}{\gamma^2} = \frac{1}{\gamma} m_0 a_{0y}$$

i.e.,

$$F_y = \frac{1}{\gamma} F_{0y} \tag{7–24}$$

In the above results one can discern the feature (mentioned in the quotation at the head of this section) that in general force and acceleration are not parallel vectors. Combining Eqs. (7–18) and (7–22) we have

$$\frac{F_y}{F_x} = \frac{1}{\gamma^2} \frac{a_y}{a_x}$$

Only in the instantaneous rest frame of a body ($\gamma = 1$) can one guarantee that $\mathbf{F}$, as defined by the time derivative of momentum, is in the same direction as the acceleration.

It is perhaps worth pointing out that the special cases of force transformation represented by Eqs. (7–20) and (7–24) can be derived in simple physical terms.[1] For the x transformation, we can consider the work done by the force, and the resulting increase of energy as manifested in a mass increase:

$$\Delta E = F_x \, \Delta x = c^2 \, \Delta m$$

where

$$\Delta x = v \, \Delta t$$

$$\Delta m = \Delta \left[\frac{m_0}{(1 - v^2/c^2)^{1/2}} \right] = \frac{m_0 v \, \Delta v}{(1 - v^2/c^2)^{3/2} c^2}$$

These at once give us

$$F_x = \gamma^3 m_0 a_x$$

which reproduces Eq. (7–18).

To relate the acceleration a_x to the acceleration a_{0x} as measured in the instantaneous rest frame of the particle, we take the equations for uniformly accelerated motion:

[1]See, for example, W. P. Ganley, *Am. J. Phys.*, **31**, 510–516 (1963) for a nice discussion.

At time t:

$$x = x_1, \qquad \text{(say)}$$

At time $t + \Delta t$:

$$x + \Delta x = x_1 + v\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

The space and time coordinates (x, t) and $(x + \Delta x, t + \Delta t)$ define two events as observed in S. Let us obtain the coordinates of these same events as observed in the rest frame S' of the particle. To do this we use the Lorentz transformations:

$$x_0 = \gamma(x - vt)$$
$$t_0 = \gamma(t - vx/c^2)$$

Applying these to the two events in turn, we have the following:

First event:

$$x_0 = \gamma(x_1 - vt)$$
$$t_0 = \gamma(t - vx_1/c^2)$$

Second event:

$$\begin{aligned} x_0 + \Delta x_0 &= \gamma[x_1 + v\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2 - v(t + \Delta t)] \\ &= \gamma[x_1 + \tfrac{1}{2}a_x(\Delta t)^2 - vt] \\ t_0 + \Delta t_0 &= \gamma[t + \Delta t - (v/c^2)\{x_1 + v\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2\}] \end{aligned}$$

Subtracting,

$$\begin{aligned} \Delta x_0 &= \gamma[\tfrac{1}{2}a_x(\Delta t)^2] \\ \Delta t_0 &= \gamma[(1 - v^2/c^2)\,\Delta t - \tfrac{1}{2}(va_x/c^2)(\Delta t)^2] \end{aligned} \tag{7–25}$$

If Δt is sufficiently short, the second term in the equation for Δt_0 becomes negligible compared to the first, and we have

$$\Delta t_0 \approx \gamma(1 - v^2/c^2)\,\Delta t = \Delta t/\gamma$$

Substituting $\Delta t = \gamma\,\Delta t_0$ in Eq. (7–25) gives us

$$\Delta x_0 = \gamma[\tfrac{1}{2}a_x(\gamma\,\Delta t_0)^2]$$

or

$$\Delta x_0 = \tfrac{1}{2}(\gamma^3 a_x)(\Delta t_0)^2$$

But this is the equation of uniformly accelerated motion for a particle initially at rest. Thus the acceleration a_{0x} as measured in the rest frame is given by $a_{0x} = \gamma^3 a_x$, which reproduces Eq. (7–19). Thus we have reproduced both Eqs. (7–18) and (7–19) and can combine them to demonstrate the invariance of F_x. This development may seem unduly long-winded, because we have carefully spelled out the transformations. It can be briefly (though somewhat glibly) summarized by saying that time dilation causes the time Δt_0 in the rest frame to correspond to $\gamma\,\Delta t_0$ in the frame S, and that Lorentz contraction causes Δx_0, the distance traveled in the rest frame in consequence of the acceleration, to correspond to $\Delta x_0/\gamma$ in S. On this basis we would arrive at once at the relation

$$\Delta x_0/\gamma = \tfrac{1}{2}a_x(\gamma\,\Delta t_0)^2$$

However, where nonproper measurements are involved, as they are here, it pays to be methodical and explicit.

To obtain the transformation of F_y, we set up Eq. (7–22) and then argue the transformation of transverse acceleration from the two statements

$$\Delta y = \tfrac{1}{2}a_y(\Delta t)^2$$
$$\Delta y' = \tfrac{1}{2}a_{0y}(\Delta t')^2$$

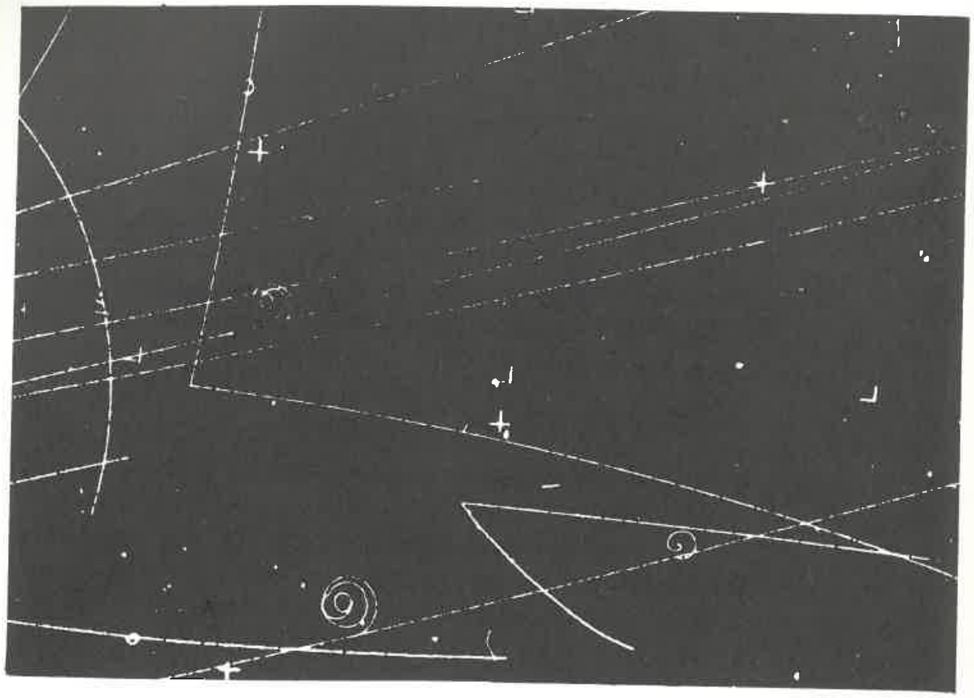
with $\Delta y = \Delta y'$ and $\Delta t = \gamma\,\Delta t'$. In this case, since the motion takes place at a constant value of x' in the rest frame, we can apply time dilation directly without any qualms.

MAGNETIC ANALYSIS OF RELATIVISTIC PARTICLES

There is one direct application of force laws in relativistic motions that we could not possibly omit from a chapter having so much to do with atomic particles. This is the deviation of a charged particle in a magnetic field. It provides one of the chief diagnostic tools in particle physics, because it reveals both the sign of the charge and the magnitude of the momentum of a particle. The bubble-chamber photograph in Fig. 7–4 is a beautiful example of the use of the technique. The basis of it is the fact that a moving charge q in a magnetic field $\mathbf{B}$ experiences a transverse force proportional to its velocity, according to the vector force law

$$\mathbf{F} = \text{const.}\,(q\mathbf{v} \times \mathbf{B})$$

In the MKS system of measurement, the value of the constant is



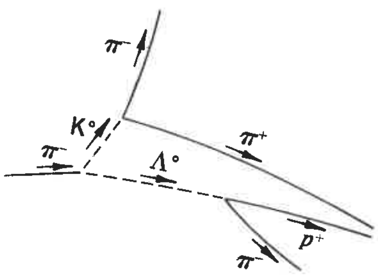


Fig. 7–4 Hydrogen bubble-chamber photograph of the production of two unstable neutral particles (K^0 and Λ^0) by the collision of a π^- meson with a proton in a 500-liter liquid hydrogen bubble chamber developed by Prof. L. W. Alvarez and his group at Berkeley. (Photo courtesy of Prof. Alvarez and Lawrence Radiation Laboratory, Berkeley, Calif.) The π^- enters from the left and its track ends abruptly, marking the point of origin of the particles K^0 and Λ^0. These subsequently decay into pairs of charged particles ($K^0 \to \pi^+ + \pi^-$, $\Lambda^0 \to \pi^- + p$), giving two pairs of forked tracks. Analysis of the tracks of the charged particles in each fork shows that the total linear momentum vector in each fork is directed outward from the point where the initial interaction took place. This picture was used as an example by Prof. D. A. Glaser, inventor of the bubble chamber, in his Nobel lecture in 1960. (See Nobel Lectures, Physics, 1942–1962, Elsevier, 1964.)

unity, by definition. If the field direction and the particle velocity are perpendicular, the motion of the particle remains in a plane perpendicular to the magnetic field. At every instant, the force exerted on the particle is at right angles to the momentum vector **p**, and the magnitude of the force is given by

$$F = qvB \qquad \text{(MKS system)}$$

Thus in a short time Δt the particle acquires a transverse momentum given by

$$\Delta p = F\,\Delta t = qvB\,\Delta t$$

This means that the momentum vector is turned through a small angle $\Delta\theta$ such that

$$\Delta\theta = \frac{\Delta p}{p} = \frac{qvB}{p}\Delta t \tag{7–26}$$

Hence the velocity vector has turned through $\Delta\theta$ also, in a time during which the particle has traveled a distance Δs such that

$$\Delta s = v\,\Delta t$$

But Δs and $\Delta\theta$ define a radius of curvature R, and in fact we have

$$\Delta\theta = \frac{\Delta s}{R} = \frac{v\,\Delta t}{R} \tag{7–27}$$

Combining Eqs. (7–26) and (7–27) we have

$$p = qBR \qquad \text{(MKS units)} \tag{7–28}$$

It is interesting to note that this is precisely the same result as one gets from Newtonian mechanics, except that in Eq. (7–28) one must remember that $p = \gamma m_0 v$, not $m_0 v$ simply. In very many cases one can be sure that $q = \pm e$, so that a knowledge of B and a measurement of the radius of curvature of the path will determine the momentum of the particle. Figure 7–4 shows a whole collection of such motions, and exemplifies the wealth of information that may be revealed by a single picture with the magnificent techniques that have been developed in particle physics.

⇒ dete. mom. of particles

GENERAL FORCE TRANSFORMATIONS; ACTION AND REACTION

In the section before last we considered some cases of force transformation that were quite special, because one of the two frames chosen was the instantaneous rest frame of the particle. But we can extend the analysis so as to yield transformations of force between any two frames, through the definitions

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \mathbf{F}' = \frac{d\mathbf{p}'}{dt'}$$

The calculation involves the use of the relativistic transformations for velocity, momentum, and energy (as well as for x and t), so for convenience we shall restate here a minimal number of the formulas needed to obtain the two basic forms of the transformation—for force components either parallel or perpendicular to the direction of relative motion of two reference frames.

Let the momentum and the energy of a particle, as measured in S, be $\mathbf{p}$ and E at the space-time point (x, y, z, t), and let $\mathbf{p}'$ and E' represent the momentum and the energy of the particle as measured in S' at the *same* space-time point (x', y', z', t'). Then we have

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ t' &= \gamma(t - vx/c^2) \end{aligned}\right\} \qquad (7\text{–}29)$$

$$\left.\begin{aligned} u_x' &= \frac{u_x - v}{1 - vu_x/c^2} \\ u_y' &= \frac{u_y/\gamma}{1 - vu_x/c^2} \end{aligned}\right\} \qquad (7\text{–}30)$$

$$\left.\begin{aligned} p_x' &= \gamma(p_x - vE/c^2) \\ p_y' &= p_y \\ E' &= \gamma(E - vp_x) \end{aligned}\right\} \qquad (7\text{–}31)$$

Since the force $\mathbf{F}'$ on a particle, as measured in S', is defined by

$$\mathbf{F}' = d\mathbf{p}'/dt'$$

it follows from equations (7–31) and the Lorentz transformation for time that

$$F_x' = \frac{dp_x'}{dt'} = \frac{\dfrac{dp_x'}{dt}}{\dfrac{dt'}{dt}} = \frac{\gamma\left(\dfrac{dp_x}{dt} - \dfrac{v}{c^2}\dfrac{dE}{dt}\right)}{\gamma\left(1 - \dfrac{v}{c^2}\dfrac{dx}{dt}\right)}$$

i.e.,

$$F_x' = \frac{F_x - (v/c^2)\,dE/dt}{1 - vu_x/c^2} \qquad (7\text{–}32)$$

Now dE/dt is the rate of change of the particle's energy as measured in S. In Newtonian mechanics we could immediately identify this with the quantity $\mathbf{F} \cdot \mathbf{u}$, the rate at which the force $\mathbf{F}$ does work. This also holds good in relativistic dynamics, as we can see by the following argument:

We have

$$E^2 = c^2p^2 + E_0{}^2 = c^2(\mathbf{p} \cdot \mathbf{p}) + E_0{}^2$$

Therefore,

$$\begin{aligned} E\,dE/dt &= c^2\mathbf{p} \cdot (d\mathbf{p}/dt) \\ &= c^2\mathbf{p} \cdot \mathbf{F} \end{aligned}$$

But $E = mc^2$. Therefore,

$$dE/dt = \mathbf{F} \cdot (\mathbf{p}/m) = \mathbf{F} \cdot \mathbf{u}$$

Hence Eq. (7–32) becomes

$$F_x' = \frac{F_x - (v/c^2)(\mathbf{F} \cdot \mathbf{u})}{1 - vu_x/c^2} \qquad (7\text{–}33)$$

Similarly (but much more simply), we find

$$F_y' = \frac{F_y}{\gamma(1 - vu_x/c^2)} \qquad (7\text{–}34)$$

(Note that, if $\mathbf{u} = 0$, we have once again the simple results $F_x' = F_x$, $F_y' = F_y/\gamma$.) An equation exactly like Eq. (7–34) holds for the relation between F_z' and F_z, and the expressions for F_x, F_y, F_z in terms of S' measurements involve writing equations equivalent to Eqs. (7–33) and (7–34) with the sign of v reversed.

F in one frame
involves [illegible]
force in other frame

Equation (7–33) is a very interesting one, in that it tells us that the measure of a force in one frame involves the measure of the *power* developed by the force in another frame. It is one more manifestation of the intermingling of space and time measurements inherent in the relativistic description of things, and has no counterpart in classical mechanics. It has been commented on as follows[1]:

> In the classical mechanics there have always been two strains of thought. The two aspects of "force" as "the time rate of change of momentum," and as "the space rate of change of energy," have with different writers been given different degrees of prominence. Galileo developed the former, Huyghens the latter. In the light of four-dimensional vectors the two ideas become unified, and differ only as partial aspects of a greater concept . . .

Pursuing this last comment, we may note that in a four-dimen

[1]By E. Cunningham, in an old but excellent book, *Relativity, The Electron Theory and Gravitation*, Longmans, Green, London, 1921.

sional space-time world, the force components F_x, F_y, and F_z represent only three of the components of some 4-vector. What is the fourth component, and what is the vector? The clue is already provided by Eq. (7–33), in which the quantity $\mathbf{F} \cdot \mathbf{u}$ appears. A simple calculation, much like that for the transformation of F_x', leads to the following result:

$$\mathbf{F}' \cdot \mathbf{u}' = \frac{(\mathbf{F} \cdot \mathbf{u}) - vF_x}{1 - vu_x/c^2} \tag{7–35}$$

The quantity $(\mathbf{F} \cdot \mathbf{u})/c$ has the same dimensions as the space components of $\mathbf{F}$ and one can, with a little extra juggling, construct an invariant from these four quantities. The exercise is a rather artificial one, however, and only serves to emphasize how momentum and energy, rather than force, provide the foundation of relativistic dynamics.

It is worth pointing out that one of Newton's basic assertions about forces between bodies—the equality of action and reaction—has almost no place in relativistic mechanics. It must essentially be a statement about the forces acting on two bodies, as a result of their mutual interaction, *at a given instant*. And, because of the relativity of simultaneity, this phrase has no unique meaning unless the points at which the forces are applied are separated by a negligible distance. It was in this sense that Max Jammer, in the remarks quoted earlier in this chapter, asserted that the concept of action at a distance has no place in relativistic dynamics. Even if the force on one object is known to be solely due to the presence of some other object, we have no unique way of describing their *mutual* interaction; we can only describe the force exerted on either body, separately, at some given point in space-time. This does *not* imply that we can no longer write down a quantitative statement of the force exerted on one body by another, as described in a given reference frame.[1] What the relativistic analysis does do, however, is to compel us to conclude that, according to measurements in a given inertial frame, the forces of action and reaction are in general *not* equal and opposite, and so the total momentum of the interacting particles is not conserved, instant by instant. This fact leads, if one wishes to hold to conservation of momentum, to the idea that momentum (as well as energy) may reside in the field that describes the interaction of separated particles. As far as the particles alone are concerned, conservation of momentum applies only when one compares the initial and final situations (before the interaction begins and after it has ceased). During the interaction itself, the momentum of the interaction field must be brought in if one is to have total momentum conservation at all times in all frames of reference.

[1] We shall, in fact, be doing this repeatedly in Chapter 8.

It is another consequence of the force transformations that if, as measured in one frame, the force on a body depends on its position but not on its velocity, then as measured in other frames the force depends on the body's velocity as well as on its position. Probably the most important example of this is in electromagnetism. The force on a moving electron due to stationary charges is given simply by Coulomb's law. But if we imagine ourselves in a frame in which the charges that caused the force are moving, then the force on the electron depends on its velocity as well as on the motion of the other charges. It involves, in fact, the magnetic force between what are effectively two currents. In this result we see the germ of the development by which electric and magnetic fields can be shown to be intimately related. To do it justice, however, this subject needs a full discussion on its own account; the hint that we have just given is clearly quite inadequate. The final chapter of this book is therefore devoted entirely to a discussion of the elements of this fascinating development.

PROBLEMS

7–1 A K meson traveling through the laboratory breaks up into two π mesons. One of the π mesons is left at rest. What was the energy of the K? What is the energy of the remaining π meson? (Rest mass of K meson = 494 MeV; rest mass of π meson $\simeq$ 137 MeV.)

7–2 An electron-positron pair can be produced by a γ ray striking a stationary electron:

$$\gamma + e^- \rightarrow e^- + e^+ + e^-$$

What is the minimum γ-ray energy that will make this process go?

7–3 Suppose that a certain accelerator can give protons a kinetic energy of 200 GeV. The rest mass m_0 of a proton is 0.938 GeV. Calculate the largest possible rest mass M_0 of a particle X that could be produced by the impact of one of these high-energy protons on a stationary proton in the following process:

$$p + p \rightarrow p + p + X$$