quite independent of any motion of the source itself. In our first discussion of the Michelson-Morley experiment (in Chapter 2) we stated that this was indeed the case. For a long time it was believed that this was proved by observations on the light from close binary stars. The two members of any such binary system have large relative velocities, and when one star has a component of velocity toward the earth the other will be moving away. It was argued that if these velocities were communicated to the emitted light, the *apparent* motions of the stars would be distorted away from the Newtonian orbits required by the law of gravitation. No such distortions were observed. It has been more recently argued, however, that since these binary star systems are usually surrounded by a gas cloud, which absorbs and then re-radiates the light from the stars, the speed of the light that crosses interstellar space may in any case be independent of any possible influence of the original moving sources.[1] Subsequently, however, experiments have been made on rapidly moving terrestrial sources of radiation which verify this aspect of Einstein's second postulate in a convincing way. In one such experiment made with high-energy photons, not visible light, the source consisted of unstable particles (neutral π mesons) traveling at 99.975% of the speed of light. The measured speed of the photons emitted forward with respect to this motion was $(2.9977 \pm 0.0004) \times 10^8$ m/sec.[2] Reference to Table 1–2 will show that this is in excellent agreement with the best values of c obtained for stationary sources. In Chapters 5 and 6 we shall discuss in more detail the radiation from moving sources, in connection with the relativistic law of addition of velocities and related phenomena.

THE RELATIVITY OF SIMULTANEITY

An immediate consequence of Einstein's prescription for synchronizing clocks at different locations is that simultaneity is relative, not absolute. Let us see how this follows.

Suppose that three observation stations A, B, and C are equally spaced along the x axis of an inertial frame S in which they are all at rest. We can construct a simple x-t coordinate system, on which we draw "world lines" (to use the accepted phraseology) showing the development of the system in space

[1]J. G. Fox, *Am. J. Phys.*, **30**, 297 (1962).

[2]T. Alväger, F. J. M. Farley, J. Kjellman and I. Wallin, *Phys. Letters*, **12**, 260 (1964).

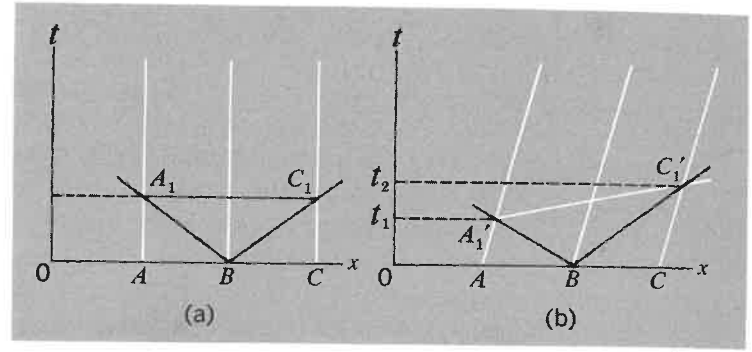

Fig. 3–2 (a) Space-time diagram showing experiment to define simultaneity at stations A and C (at rest in this reference frame) by light signals emitted from a station B midway between them. (b) Equivalent experiment for the case in which A, B, and C all have a velocity with respect to the reference frame.

and time [Fig. 3–2(a)]. The world line of any given particle is just a graph of its position as a function of time; it provides a complete picture of the history of the particle as observed within a given frame of reference. The world lines of A, B, and C are of course just vertical lines parallel to the t axis, corresponding to $x =$ constant. Suppose that a light or radio signal is sent out from B at $t = 0$. It travels at the same speed c forward and backward along the x axis—an assertion that embodies the universality of c. This signal is described by two sloping lines $x = x_B \pm ct$. The arrival of the signal at the positions of A and C is thus given by the intersections A_1, C_1, and simultaneity at the positions of A and C is *defined* by the line A_1C_1, parallel to the x axis, which joins a series of points possessing the same value of t.

But now suppose that A, B, and C are at rest in an inertial frame S' which is moving with respect to S at a speed v along the x direction [Fig. 3–2(b)]. The world lines of A, B, and C are now inclined as shown. A signal sent from B at $t = 0$ is again described (in S) by the lines $x = x_B \pm ct$, and the arrival of the signal at the positions of A and C is now given by the intersections A_1' and C_1'. These are clearly not simultaneous for S, because the line $A_1'C_1'$ is manifestly not parallel to the x axis. Or, to put it more concretely, the signal reaches A before it reaches C because, as observed in S, A is running to meet the signal pulse whereas C is running away from it. But we require

B to be midway between A and C in S' as well as in S. Accepting the universality of c and the equivalence of inertial frames, we therefore *demand* that A_1' and C_1' represent simultaneous events in S'. (An *event*, from the standpoint of relativity theory, is completely characterized by its space and time coordinates in a given frame of reference.) *Our judgment of simultaneity is a function of the particular frame of reference we use.*

It is natural to ask why we should base this definition of simultaneity on the velocity c in particular and not on any other possible signal velocity. The simplest answer to this is to point to the obvious uniqueness of c, not merely as the speed of light, but as the ultimate speed in all of dynamics. A more convincing reply (at least in the long run) is that this choice does indeed have the consequence that every known physical law has the same form in all inertial frames.

In the above discussion we have demonstrated in a qualitative way the relativity of simultaneity. Our next step must be to develop the quantitative aspects of time and space measurements according to special relativity.

THE LORENTZ-EINSTEIN TRANSFORMATIONS

Look now at Fig. 3–3. It depicts the operation of defining simultaneity at stations A and C which are moving at speed v with respect to an inertial frame S. We have already discussed such a diagram (cf. Fig. 3–2). But now we have added lines to represent the coordinate axes of the frame S' in which A and C

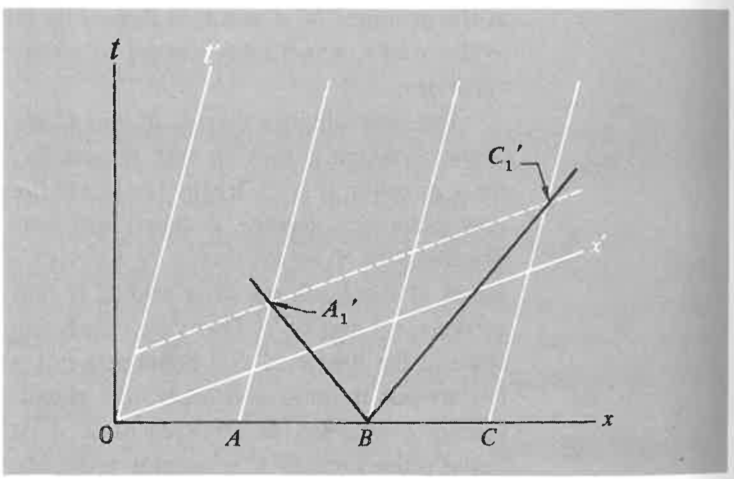

Fig. 3–3 Specification of coordinate axes (x, t) and (x', t') for two reference frames in relative motion.

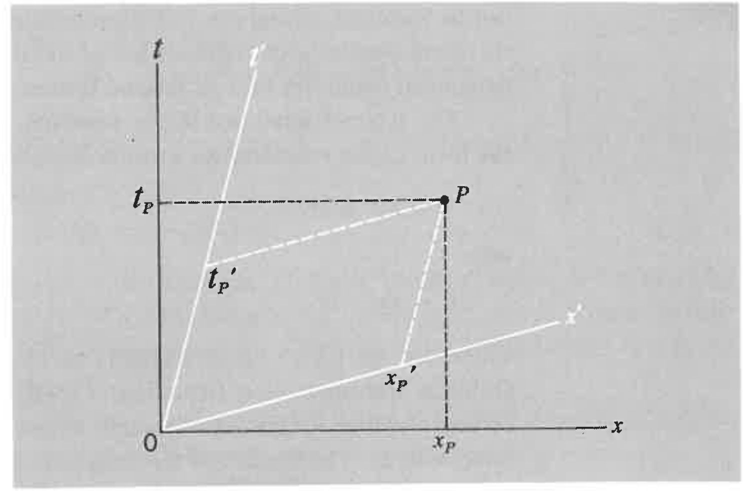

Fig. 3–4 Space-time coordinates of a given point event in two different inertial frames.

introducing the 2nd coord. sys. at rel. speed

are at rest. How have we done this? The axis of t' is readily described; it is the line $x' = 0$, i.e., the world line of the origin of S'. And since the frame S' has a speed v along the x direction with respect to S, the position of this origin is described in S by the equation $x = vt$ if the origins of S and S' coincide at the time $t = 0$.

What about the axis of x'? This is the line that connects all points corresponding to $t' = 0$. Any line of the form $t' =$ constant is parallel to this x' axis. But the line $A_1'C_1'$ is just such a line, since A_1' and C_1' are events by which simultaneity in S' is defined. Hence we construct the x' axis by drawing a line parallel to $A_1'C_1'$, and for convenience we make it pass through O, which is thus described both by $x = 0$, $t = 0$ and by $x' = 0$, $t' = 0$. The noncoincidence of the axes of x and x' does not, of course, imply any geometric tilting of one with respect to the other; it is a purely formal tilting in the abstract space constructed from the x and t coordinates.

Now this type of diagram displays for us a key feature of the kinematic transformations of special relativity. In Fig. 3–4 any point P in the plane of the diagram represents what is called a *point event*, which can be characterized alternatively by the values of x and t or of x' and t'. And our construction implies that x' and t' alike should be linear functions of both x and t. Similarly, x and t are linear functions of x' and t'. This linearity is a fundamental property of the transformation equations. If they did not have this form, a motion recorded as motion at constant velocity along a straight line in one frame (say S) would not be recorded as uniform rectilinear motion in S'. This would therefore conflict with Galileo's law of inertia and with our basic dynamical condition that all inertial frames are equivalent.

The symmetry implied by the relativity principle means that the form of the relationships must be as follows:

$$x = ax' + bt'$$

with

$$x' = ax - bt \tag{3–8}$$

These are set up so as to resemble as closely as possible the Galilean transformation [equations (3–4)] to which they must certainly reduce for sufficiently small values of the speed v of S' relative to S. The motion of the origin of S as measured in S' is defined by putting $x = 0$ in the first of these equations. Similarly, the motion of the origin of S' as measured in S is defined by putting $x' = 0$ in the second equation. The velocities are equal and opposite and both of magnitude v. This gives us the condition

$$b/a = v \tag{3–9}$$

Next we consider the descriptions according to S and S' of a light signal traveling in the positive x direction. Let the signal originate at O of Fig. 3–3. It is then described by the following very simple equations in S and S', respectively:

$$x = ct \qquad x' = ct' \tag{3–10}$$

Substitute these particular expressions for x and x' in equations (3–8), and we get the following:

$$\begin{aligned} ct &= (ac + b)t' \\ ct' &= (ac - b)t \end{aligned} \tag{3–11}$$

Eliminating t and t' between these last equations, and using the condition $b = av$ from Eq. (3–9), we find

$$c^2 = a^2(c^2 - v^2)$$

Therefore,

$$a = \frac{1}{(1 - v^2/c^2)^{1/2}} \tag{3–12}$$

It may be noted that this coefficient, a, is precisely the factor $\gamma(v)$ that emerged in our dynamical analysis in Chapter 1—cf. Eq. (1–22). We can now rewrite equations (3–8) in the following explicit form:

$$\left.\begin{aligned} x &= \frac{1}{(1 - v^2/c^2)^{1/2}}(x' + vt') = \gamma(x' + vt') \\ \text{and} \quad x' &= \frac{1}{(1 - v^2/c^2)^{1/2}}(x - vt) = \gamma(x - vt) \\ \text{where} \quad \gamma(v) &= (1 - v^2/c^2)^{-1/2} \end{aligned}\right\} \tag{3–13}$$

These differ from the Galilean transformations by having the factor γ (≥ 1) as a multiplier on the right, and it is clear that the Galilean equations are a limiting form of equations (3–13) for $v/c \to 0$.

Given equations (3–13) it is a matter of elementary algebra to obtain the following expressions for t and t':

$$\begin{aligned} t &= \gamma(t' + vx'/c^2) \\ t' &= \gamma(t - vx/c^2) \end{aligned} \tag{3–14}$$

The reduction of these to the Galilean relation $t = t'$ requires $x \ll ct$ as well as $v/c \ll 1$. Equations (3–13) and (3–14) are the revised version of the transformations relating x and t for two inertial frames in relative motion along the x direction.

To complete our statement of the transformation of measurements between two reference frames in relative motion, we need the connection between measures of a distance (y or z) transverse to the direction of relative motion of the frames. Clearly, if space is isotropic, all displacements transverse to the unique direction defined by the relative motion are equivalent, and it is not difficult to conclude that the appropriate transformations are of simple equality:

$$y = y' \qquad z = z' \tag{3–15}$$

We can argue this on the grounds that, if it were *not* true, we should have a way of detecting absolute displacements and motions. Imagine, for example, that we constructed two identical square grids, representing xy coordinate systems, one of which is going to be associated with a frame S and the other with a frame S'. We assume that the grids are first checked for identity of spacing when at rest relative to each other. We suppose that they are then set in relative motion along x, and that paint brushes mounted every 10 cm along the y axis of S leave stripes on the grid S'. Likewise, paint brushes mounted every 10 cm along y' leave stripes on S. Each inertial frame thus receives its explicit form:

$$\left.\begin{aligned} x &= \frac{1}{(1 - v^2/c^2)^{1/2}}(x' + vt') = \gamma(x' + vt') \\ \text{and} \quad x' &= \frac{1}{(1 - v^2/c^2)^{1/2}}(x - vt) = \gamma(x - vt) \\ \text{where} \quad & \\ \gamma(v) &= (1 - v^2/c^2)^{-1/2} \end{aligned}\right\} \qquad (3\text{–}13)$$

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The above array of transformation equations was introduced by H. A. Lorentz in 1904 as a basis for modifying electromagnetic theory so as to reconcile the null result of the Michelson-Morley experiment with the existence of a unique inertial frame provided by the luminiferous ether. But Einstein discovered the equations quite independently a year later with the help of his fresh and radical approach to the whole problem.

MORE ABOUT THE LORENTZ TRANSFORMATIONS[1]

In deriving the Lorentz transformations in the last section, we considered only the requirements imposed by light signals traveling along the x direction. A more general approach would have developed them by applying the requirements of Einstein's second postulate to a light signal traveling in an arbitrary direction. Having already set up the transformations, however, we can use them to illustrate a seeming paradox which is contained in Einstein's postulate. It is this: Suppose that a burst of light begins spreading out (in vacuum) from the origin of frame S at $t = 0$. At any later time t the light will have reached all points on a sphere of radius r, centered on the origin of S, such that $r = ct$. Then if this same phenomenon is observed with respect to a frame S', moving with respect to S with any velocity v, the description of the expanding burst of light is again a sphere, in this case centered on the origin of S'—even though, by definition, the origins of S and S' coincide only at the instant $t = t' = 0$.

To see how this result emerges, we take the equation $r = ct$ and rewrite it in terms of position and time coordinates measured in S'. By first squaring both sides of the equation we get

$$r^2 = x^2 + y^2 + z^2 = c^2t^2$$

Now use the right-hand set of equations (3–16). The above equation then becomes the following:

$$\gamma^2(x' + vt')^2 + (y')^2 + (z')^2 = \gamma^2c^2(t' + vx'/c^2)^2$$

It may be noted that the cross terms in $x't'$ on the two sides of the equation are equal, and so disappear. Collecting the other terms, we have

$$\gamma^2(x')^2(1 - v^2/c^2) + (y')^2 + (z')^2 = \gamma^2(t')^2(c^2 - v^2)$$

[1] Having once recognized that these transformations were arrived at by both Lorentz and Einstein, we shall usually in future refer to them by this briefer and more customary title.

But

$$\gamma^2(1 - v^2/c^2) = 1$$

Therefore,

$$(x')^2 + (y')^2 + (z')^2 = c^2(t')^2$$

which defines a sphere of radius r' such that

$$r' = ct'$$

This result, which at first sight appears to do violence to one's commonsense ideas, is bound up with the relativity of simultaneity. Points which, as measured in S, are reached at the *same* time t, are reached at *different* times as measured in S', in such a fashion that the light is properly described as lying on a spherical shell expanding at speed c in both frames.

MINKOWSKI DIAGRAMS: SPACE-TIME

A valuable aid to the arguments in this chapter has been the use of graphs, with axes representing position and time, which allow one to display the complete history of a one-dimensional motion. The use of such graphs in special relativity was introduced by H. Minkowski in 1908, and they are customarily referred to as Minkowski diagrams. On any such diagram, as we have seen, any individual event—e.g., a light signal striking a detector, or one tick of a watch—is uniquely represented by some point P (Fig. 3–6). The detailed specification of this event, however, in terms of numerical values of x and t, can be made in infinitely many different ways according to the particular reference frame chosen. The description of a point event is described in frame S by the coordinates (x, t) and in S' by the coordinates (x', t'). If the origins of S and S' are chosen so as to coincide at $t = t' = 0$, then the relation between (x, t) and (x', t') is contained in the Lorentz transformations of equations (3–16).

It is very convenient to use ct, rather than t, to describe the time coordinate. Both coordinates, ct and x, are then expressed as distances, and if the scale of distance is chosen to be the same for both, the world line of a light signal starting out at $x = 0$, $t = 0$, is a bisector of the angle between the axes. This holds good in all reference frames. We can represent any one such frame (say, S) by drawing the axes of x and ct at right angles to one another. Other reference frames (S' and S'', for example) are then characterized by nonorthogonal axes for distance and time. (In Fig. 3–6, S' has some positive velocity along x with respect to S, and S'' has a negative velocity.) There is nothing special or privileged about that frame which we choose to show with orthogonal space and time axes.

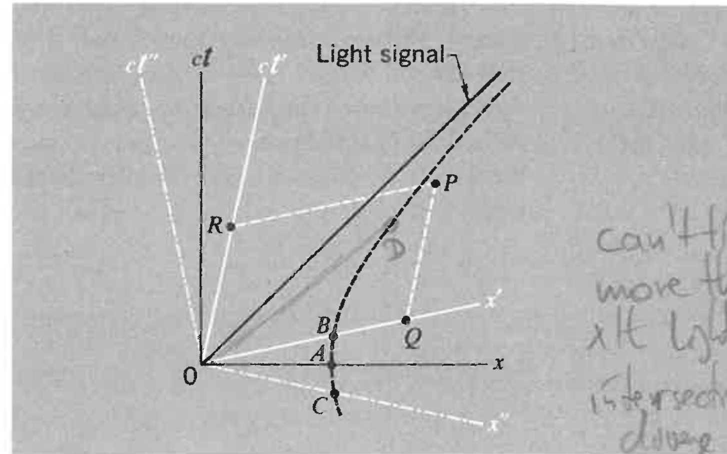

Fig. 3-6 Minkowski diagram, showing three different coordinate systems and a calibration hyperbola to define unit distance along x for each system.

To read off the space and time coordinates of a given point event P, we draw lines through it, parallel to the space and time axes of any chosen reference frame, and read off the intercepts. In Fig. 3–6 we have taken S' as the frame, so OQ gives the measure of x' and OR the measure of ct'. It is very important to realize, however, that in a diagram such as Fig. 3–6, *the scale, representing unit distance, is not the same along the different axes x, x', x'', etc.* To be specific, if we draw the rectangular hyperbola defined by

$$x^2 - (ct)^2 = 1 \tag{3–17}$$

its intersections A, B, and C with the axes of x, x', and x'' will in each case define unit distance from O for the particular frame in question (see Fig. 3–6). The justification of this procedure is embodied in the discussion in the next section.

A SPACE-TIME INVARIANT

Let us now return to the basic kinematic relations of special relativity, as expressed in the Lorentz transformations. We have seen how special relativity was born out of Einstein's recognition