Low-pass filters for signal averaging

Edward Voigtman and James D. Winefordner

Citation: **57**, 957 (1986); doi: 10.1063/1.1138645
View online: http://dx.doi.org/10.1063/1.1138645
View Table of Contents: http://aip.scitation.org/toc/rsi/57/5
Published by the American Institute of Physics

Low-pass filters for signal averaging

Edward Voigtman and James D. Winefordner[a)]

Chemistry Department, University of Florida, Gainesville, Florida 32611

(Received 20 May 1985; accepted for publication 12 January 1986)

Detailed comparison of the settling time-noise bandwidth products of 31 types of low-pass filters demonstrates that the settling time-noise bandwidth product is the figure of merit for such filters when the goal is averaging. Common filters such as Butterworth, elliptic, and Chebyshev are found to be unusable for such purposes while others, such as Bessel filters, offer only moderate figures of merit. The best reported analog low-pass filter differs from ideality by only about 11%. The optimum analog low-pass filter, having continuous, rational transfer function, is unknown.

INTRODUCTION

Consider the problem of measuring a slowly varying, nonzero mean signal in additive, white noise. The signal may be the output of a previous demodulation stage so the mean is nonzero by contrivance.[1] Thus, the signal power spectrum is narrow, centered on dc, and, in the limiting case of a dc signal, is just a "delta function" at zero frequency, i.e., the dc signal is fully characterized by its mean value (signal power spectral component at $\omega = 0$). In contrast, the white noise power spectrum is constant, with bilateral power spectral density of η W/Hz, so that signal and noise power spectral overlap is small or negligible. In terms of time autocorrelation functions,[2] the signal autocorrelation function is very broad or constant (for dc) so that the signal is easily predictable. The noise autocorrelation function is a delta function so that no predictability is possible. The dissimilarity between the signal and noise power spectra (and time autocorrelation functions) is so great that signal processing is relatively easy. Optimum, linear signal processing results in optimum power signal-to-noise ratio (S/N), but at the expense of a mean square error of[3]

$$\overline{e^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S_x(\omega)S_n(\omega)}{S_x(\omega)+S_n(\omega)}\,d\omega, \tag{1}$$

where $S_x(\omega)$ is the signal power spectrum and $S_n(\omega)$ is the noise power spectrum. The optimum filter power transfer function (frequency domain power response function) is[4]

$$\left|H(\omega)\right|^2 = \frac{S_x(\omega)}{S_x(\omega)+S_n(\omega)}, \tag{2}$$

which may not be physically realizable. Evidently, the optimum filter is a low-pass filter which most heavily weights spectral regions of high S/N. In the dc limit, i.e., $S_x(\omega) = \delta(\omega)$ and $S_n(\omega) = \eta$, continuous integration is optimum and the error in the signal estimate may be arbitrarily reduced at the expense of measurement time.[5] As will be seen, the optimum filter has zero bandwidth, but nonzero power transfer function.

Even in the case of a slowly varying signal, it is easier to accurately and precisely estimate the signal mean, than it is to obtain a low-noise, low-distortion (small error) temporal copy of the signal. Normally, averaging is performed with low-order ($n < 5$), low-pass filters and distortion is irrelevant because all of the ac signal spectral power is deliberately sacrificed. Where low distortion is a constraint, higher-order filters with sharp cutoff, e.g., fourth-order Butterworth or Chebyshev, are preferred.[6]

It should also be noted that phase linearity is an important criterion if slowly varying signals, e.g., scanned spectra or chromatograms, are to be processed with minimum phase distortion. In this case, the filter should have linear phase (equivalent to symmetric impulse and step responses). Since Bessel filters are the maximally flat phase approximations to the ideal time domain (Gaussian) filters, they are good overall choices where both dc and low ac signal frequencies must be processed. We will emphasize the dc case, i.e., the signal varies little over the measurement period, because it is difficult to know, by inspection of a given low-pass filter (LPF) impulse response, whether it is sufficiently symmetric to imply good phase linearity. Indeed, the impulse response of the third-order Bessel LPF does not seem particularly symmetric to the unpracticed eye.

However, LPF averaging is of more importance in noise reduction efforts than is generally realized since linear, time-invariant filters also determine the noise reduction behavior of linear, time-variant filters such as lock-in amplifiers, boxcar integrators/averagers, signal averagers, and correlators. Curiously, the figure of merit for LPF averaging filters has escaped notice. It is the intention of this paper to exhibit the natural figure of merit for LPF averaging applications and show that the best reported such filters depart from ideality by only about 11%.

Now suppose the signal, denoted by $x(t)$, is applied to the input of a stable, linear, time-invariant, low-pass filter with causal impulse response $h(t)$. Then the time domain output is the convolution[7] of $x(t)$ and $h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau. \tag{3}$$

If $x(t)$ is also causal, the lower integration limit is 0 and the measurement (integration) time is finite. The mean or expected value of $x(t)$, denoted by $E[x(t)]$, is[8]

$$E[x(t)] \equiv \lim_{t\to\infty}\frac{1}{t}\int_0^t x(\tau)d\tau. \tag{4}$$

Comparison of Eqs. (3) and (4) shows that the low-pass filter output well approximates the input mean, to within a scale factor, if the measurement time is prolonged and $h(t)$ is almost constant over the measurement time. For the limiting case of causal dc input, e.g., a unit step input, the LPF output is the unit step response

$$y(t)=\int_{-\infty}^{\infty}U(\tau)h(t-\tau)d\tau=\int_{0}^{t}h(t-\tau)d\tau=\int_{0}^{t}h(\tau)d\tau, \tag{5}$$

where $U(t)$ is the unit step function and $y(t)$ asymptotically approaches $H(0)$ (the dc gain).

Thus, for a given measurement period, the best LPF for estimating a signal mean is the LPF which has constant impulse response over the measurement period and zero impulse response elsewhere. In this case, the impulse response may be removed from the integral in Eq. (3) and the integration limits are finite, so the signal is merely integrated over the measurement time. Note that the abrupt zeroing of the impulse response at the end of the measurement period gives the gated integrator its perfect settling characteristic.

The optimality of the stable integrators may be confirmed by noting that the optimum processing, in least-squares sense and for maximum power S/N ratio, is well known to be matched filtering, i.e., low-pass filtering with impulse response equal to a time-reversed, delayed replica of the input signal.[9] Thus, the matched filter for the step function input may be approximated with a gated integrator having rectangular pulse response. Note that this is a truncated (causal) version of the actual acausal impulse response, which would be a time-reversed step function at t. This example points out the major problem with matched filters: they are usually not physically realizable. Fortunately, they possess one remarkable advantage: it is extraordinarily easy to compute the optimum power S/N obtained with matched filtering since it is simply[9]

$$S/N=E/\eta, \tag{6}$$

where E is the signal energy (J) and η is the bilateral noise power density (W/Hz). Thus the matched filter is the standard against which all other filters can be judged. For power signals, the energy is

$$E(t)=\int_{0}^{t}P(\tau)d\tau, \tag{7}$$

which reduces to a product if the power is constant, as for a step function signal. Thus, the power S/N increases with measurement time, as expected.

Alternatively, the power S/N can be calculated directly since the mean signal power is given by Eq. (4), with $x(t)$ the instantaneous signal power. The noise power is given by

$$P_n=\int_{-\infty}^{\infty}\eta|H(f)|^2df=\int_{0}^{\infty}2\eta|H(f)|^2df, \tag{8}$$

where $|H(f)|^2[\equiv H^*(f)H(f)]$ is the filter power transfer function. For white noise, the noise power density is constant, so it may be removed from the noise power integral, yielding

$$P_n=2\eta|H(f)|^2_{\max}\left(\frac{1}{|H(f)|^2_{\max}}\int_{0}^{\infty}|H(f)|^2df\right). \tag{9}$$

Thus, the noise power is the product of the constant (unilateral) noise power density, the maximum power gain, and the (equivalent) noise bandwidth. Hence, the noise bandwidth is defined as

$$\frac{\omega_n}{2\pi}\equiv B_n\equiv\frac{1}{2\pi}\frac{1}{|H(\omega)|^2_{\max}}\int_{0}^{\infty}|H(\omega)|^2d\omega \tag{10}$$

and is simply the bandwidth an ideal ("brickwall") LPF would have if the ideal filter had equal maximum gain and equal area under its power transfer function. As an example, consider the gated integrator transfer function

$$H(f)=\frac{\tau_a}{\tau_i}\operatorname{sinc}f\tau_a e^{-i2\pi f\tau_a/2}=\frac{\tau_a}{\tau_i}\operatorname{sinc}f\tau_a e^{-i\omega\tau_a/2}, \tag{11}$$

where $\operatorname{sinc}(x)\equiv\sin(\pi x)/\pi x$, τ_a is the aperture (integration) time, and τ_i is the integration time constant. Then

$$|H(f)|^2\equiv H^*(f)H(f)=(\tau_a/\tau_i)^2\operatorname{sinc}^2 f\tau_a, \tag{12}$$

with maximum power gain at dc. Substituting Eq. (12) into (10) yields $B_n=1/2\tau_a$. Since signal power and noise power density are equally affected by the maximum power gain of $(\tau_a/\tau_i)^2$, it follows that the power S/N increases with τ_a, as before. Note that although B_n approaches 0 as τ_a goes to infinity (the continuous integrator limit), the product of the dc power gain and noise bandwidth increases as τ_a, so that noise power increases as τ_a. Signal power increases as τ_a^2, so the power S/N increases as τ_a.

However, the continuous integrator has a pole at the origin and, therefore, is only marginally stable. The gated and running integrators have optimal impulse response and exactly settle at the end of the integration period,[10] but the (stable) transform is transcendental. Since the transfer functions are not rational fractions, they cannot be implemented with time-invariant, lumped parameter circuits.[11]

Unfortunately, it is difficult or cumbersome to implement integration LPFs because continuous integrators may give unbounded output for bounded input, gated integrators require external timing for readout and reset, and running integrators are impractical for audio and subaudio frequencies because of the difficulty in implementing long analog delays. Serial analog delay (SAD) lines and transversal filters[12] do not provide a wholly satisfactory solution to the running integrator design problem, although it is quite easy to devise a digital running integrator, e.g., the Oriel 76000 signal processor. However, such digital filters must be preceded by analog antialiasing filters,[13] unless either the signal and noise are already bandlimited[14] (so that the white-noise assumption is violated and a preceding stage is serving as a de facto antialiasing filter), or the S/N is so high that the noise is below the digitization error (quantization noise) of the succeeding A/D converter, in which case filtering is unnecessary. Oversampling, i.e., use of sampling rates much in excess of the Nyquist rate, avoids the aliasing problem, but may not always be possible because of hardware or software restrictions.

Thus, antialiasing filters are always necessary. Since noise aliasing cannot be removed by clever digital postprocessing, there can be no digital Lazarus filter, i.e., a digital resurrection of a signal mortally harmed by inept analog preprocessing. As noted above, it is also harder to design a filter which removes noise without causing signal distortion[15] and the antialiasing filter is particularly hard to design since it must be linear phase and have superb frequency domain characteristics.[16] Thus, it is easier to design a good analog LPF than it is to design a good digital filter with proper antialiasing filter. Any meaningful discussion of the relative merits of analog versus digital filters must address the antialiasing filter problem.

I. LOW-PASS FILTER FIGURES OF MERIT

Given the difficulties in using integrators (continuous, gated, or running) as averaging LPFs, it is natural to wonder how well an ordinary *RC* LPF would perform. For the case of a rectangular signal pulse of duration τ_a, it is well known that the optimum *RC* value is 0.7959 τ_a and the (time-dependent) optimum power S/N is 81.45% of that attainable with matched filtering.[17] However, the *RC* LPF is primitive, offering poor performance in both the time and frequency domains. It is known, for example, that a good LPF is more more than twice as fast as the *RC* LPF for equal noise bandwidths.[10] Note that it is unfair to compare LPFs of different type or order on the basis of arbitrary measures such as equal signal bandwidths or time constants because, with equal white-noise power density inputs, the mean-square output fluctuations would be unequal. In other words, the mean output noise powers would be different. Furthermore, the impulse response of a real LPF cannot abruptly change, as in the gated integrator, so the unit step response is only approached asymptotically. Hence, the LPF output, for dc input, is not a simple ramp (sometimes mistakenly said to be "linear" output[18]) except, as Eq. (5) shows, in the integration case.

There are two conditions to be satisfied for a fair comparison of LPFs. First, the LPFs must have equal dc gains

$$H(0) = \int_{-\infty}^{\infty} h(t)dt = \int_{0}^{\infty} h(t)dt$$

$$= \lim_{t\to\infty}\int_0^t U(t-\tau)h(\tau)d\tau, \qquad (13)$$

where $h(t)$ is causal. The first equality follows from the definition of the Fourier transform, with $\omega = 0$. It is convenient to let $H(0) = 1$, which is equivalent to specifying normalized impulse responses and equal ultimate unit step responses.

However, it is not possible to wait forever for settling to occur, so a settling tolerance must be adopted. As in the case for operational amplifiers, we define the settling time as the time required for the filter output to first reach and stay within a specified symmetrical error band about the ultimate (steady-state) value of unity, when the input is a unit step function at $t = 0$. An arbitrary, but reasonable error tolerance is $\pm$ 1% since this corresponds to the common practice of taking measurements every 4–5 time constants. For a first-order *RC* LPF, 1% settling occurs at about 4.605 *RC*. An error tolerance smaller than 1% gives better measurement resolution at the expense of longer measurement time, while larger error tolerances trade accuracy for speed. Thus, the choice is arbitrary. This will be discussed in more detail later. The important point is that a filter with small time constant settles quickly, but does not reduce noise very well. Therefore, another condition is necessary.

The second condition is that the LPFs must have equal impulse response energies

$$E_I = \int_{-\infty}^{\infty} h^2(t)dt = \int_0^{\infty} h^2(t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{\infty} |H(\omega)|^2 d\omega, \qquad (14)$$

$$= |H(\omega)|^2_{\max} B_s = 2|H(\omega)|^2_{\max} B_n = R_h(0), \qquad (15)$$

where E_I is the impulse response energy,[19] B_s is the spectral bandwidth, and $R_h(0)$ is the autocorrelation function at zero delay. Parseval's theorem provides the connection between E_I and B_s. Note that B_s is the bilateral version of B_n. The equivalent bilateral notion for the LPF time autocorrelation function is the autocorrelation time, i.e., the autocorrelation time is the integral from $-\infty$ to $+\infty$ of the autocorrelation function with respect to τ, divided by $R_h(0)$. For all LPF filters without passband ripple, the product of the spectral bandwidth and autocorrelation time is unity[20] because the maximum power gain is at dc, so it is convenient to specify

$$H(0) \equiv 1 \equiv H_{\max}(\omega). \qquad (16)$$

Thus, the second condition is equivalent to specifying that LPFs to be compared must have equal noise bandwidths, i.e., for equal white-noise power densities as inputs, they have equal output noise powers.

Therefore, a satisfactory figure of merit for low-pass filters for averaging purposes is the 1% settling time-noise bandwidth product, which must be minimized. The gated integrator has the best figure of merit.[10] Other filters can merely approach its product of 0.495 ($= 0.99\,\tau_a \times 1/2\tau_a$).

As previously mentioned, other settling time criteria can be adopted including settling to 0.1% and settling to the A/D rms quantization noise.[21] Other "bandwidth" criteria include 1-Hz or 1-rad/s spectral bandwidth, 1-rad/s noise bandwidth, or 1-s autocorrelation time. The most convenient choice for normalization purposes is 1-Hz noise bandwidth. Therefore, LPFs normalized to 1-Hz noise bandwidth may be fairly compared on the basis of 1% settling times if it is recognized that changes in the settling tolerance may drastically alter the relative performances of various LPFs under consideration. This will be discussed later. Note, however, that ordinary (arbitrary) engineering measures such as "rise time," "FWHM," signal bandwidth, and so on, are of no intrinsic merit and cannot be used to fairly compare filter performance. Indeed, the use of such measures may lead to seemingly contradictory comparisons of filters.[22] It is essential that a figure of merit be valid; it is not essential that it be easily defined or measured. In the present case, the figure of merit is all three.

II. ANALYTICAL RESULTS

All cascaded, nth order filters are assumed to consist of n, buffered, equal time constant, first-order filters. Noise bandwidths are obtained from Eq. (10) with the appropriate power transfer function while step responses are obtained by integration [Eq. (5)] of the impulse responses obtained by inverse transformation of the nth order transfer functions.

The choise of filters to be examined is determined by importance, usefulness, and tractability. The synchronous LPF is both primitive and ubiquitous and is found in virtually all commercial lock-in amplifiers and boxcar averagers. The cascaded, gated integrator LPF is analytically tractable while the "brickwall" and Gaussian LPFs are the ideal frequency and time domain LPFs. The latter two are examined with their acausal step responses further idealized, i.e., piecewise linearly approximated in accordance with Su's criterion.[23] As will be seen, the ideal time domain filter is better than the ideal frequency domain filter, for averaging purposes, by about 50%. Since ideal filters cannot be constructed, their figures of merit serve as benchmarks against which actual LPF performance can be gauged.

CASCADED RC LPFs: The transfer function for the nth order synchronous filter is, where $\tau \equiv RC$,

$$H(p) = H(s\tau) = \frac{1}{(1+s\tau)^n} = \tau^{-n}\frac{1}{(s+1/\tau)^n}. \quad (17)$$

The inverse Laplace transform of Eq. (17) yields

$$h(t) = \tau^{-n}\frac{t^{n-1}}{\Gamma(n)}e^{-t/\tau}U(t), \quad (18)$$

where $\Gamma(n)$ is the gamma function. Substitution of Eq. (18) into (5), and simplification, gives the step response

$$y(t) = 1 - e^{-t/\tau}\sum_{r=0}^{n-1}\frac{1}{r!}\left(\frac{t}{\tau}\right)^r. \quad (19)$$

From Eq. (17), the noise bandwidth is

$$B_n = \frac{a^{2n}}{2\pi}\int_0^\infty \frac{d\omega}{(\omega^2+a^2)^n}, \quad (20)$$

where $a \equiv 1/\tau$ and the integral in Eq. (20) is[24]

$$\int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2(2n-2)!!}\frac{\pi}{a^{2n-1}}, \quad (21)$$

with $(n+1)!! \equiv (n+1)(n-1)(n-3)...(1)$ and $0!! = 1$. Then the following relations[25]

$$(2n)!! = 2^n n! = 2n(2n-2)!!, \quad (22)$$

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!} = (2n+1)(2n-1)(2n-3)!!, \quad (23)$$

$$(-1)!! = 1, \quad (24)$$

together with Eqs. (20) and (21) yield

$$B_n = \frac{(2n)!}{2^{2n+1}(2n-1)(n-1)!n!RC} = \frac{(2n-2)!}{2^{2n}[(n-1)!]^2RC}. \quad (25)$$

Slightly incorrect B_n expressions are given in several references[26,27] while a completely wrong expression is given by Moore *et al.*[28] The signal bandwidth (-3 dB) is obtained by setting the power transfer function equal to $\frac{1}{2}$, giving

$$B \equiv f_{-3\,\mathrm{dB}} = (2^{1/n}-1)^{1/2}/2\pi RC. \quad (26)$$

The bandwidth of a cascade of filters is often incorrectly given. It is easy to show that

$$\lim_{n\to\infty}\frac{B_n}{B} = \frac{1}{2}\left(\frac{\pi}{\ln 2}\right)^{1/2} \cong 1.064467. \quad (27)$$

CASCADED, SYNCHRONIZED, GATED INTEGRATORS: The transfer function of n cascaded, synchronized, gated integrators, with equal τ_a and τ_i values, is

$$H(\omega) = \left(\frac{\tau_a}{\tau_i}\right)^n \mathrm{sinc}^n f\tau_a e^{-i\omega n\tau_a/2}. \quad (28)$$

The impulse response is the inverse Fourier transform of Eq. (28)

$$h(t) = \frac{1}{2\pi}\left(\frac{\tau_a}{\tau_i}\right)^n\int_{-\infty}^{\infty}\left(\frac{\sin \omega\tau_a/2}{\omega\tau_a/2}\right)^n e^{-i\omega n\tau_a/2}e^{i\omega t}d\omega, \quad (29)$$

since $H(\omega)$ is stable and $s = i\omega$. Assuming unity dc gain and using $\exp(-ix) = \cos(x) - i\sin(x)$ gives

$$h(t) = \frac{1}{2\pi}\frac{2}{\tau_a}\int_{-\infty}^{\infty}\left(\frac{\sin\omega\tau_a/2}{\omega\tau_a/2}\right)^n \times\left[\cos\omega\left(\frac{n\tau_a}{2}-t\right) - i\sin\omega\left(\frac{n\tau_a}{2}-t\right)\right]d\left(\frac{\omega\tau_a}{2}\right). \quad (30)$$

But the first factor in the integrand is even with respect to ω while the cosine term is also even and the sine term is odd. Thus

$$h(t) = \frac{2}{\pi\tau_a}\int_0^\infty\left(\frac{\sin\omega\tau_a/2}{\omega\tau_a/2}\right)^n\cos\omega\left(\frac{n\tau_a}{2}-t\right)d\left(\frac{\omega\tau_a}{2}\right). \quad (31)$$

Note that $\cos[\omega(n\tau_a/2 - t)]$ is also even about $t = n\tau_a/2$. Then, using the tabulated integral,[29] the result for $0 < t \leqslant n\tau_a/2$ is

$$h(t) = \frac{n}{2^{n-1}\tau_a} \times \sum_{k=0}^{k<(n+m)/2}\frac{(-1)^k[2(n-k-t/\tau_a)]^{n-1}}{k!(n-k)!}. \quad (32)$$

Since $h(t)$ is symmetrical about $t = n\tau_a/2$, causality implies that $h(t) = 0$ for $t > n\tau_a$. Thus the unit step response is

$$y(t) = \int_0^{t\leqslant n\tau_a} h(\tau)d\tau, \quad (33)$$

with $y(n\tau_a) = 1$. Another implication of the symmetry of $h(t)$ about $t = n\tau_a/2$ follows from the observation that cascaded, noninverting integrators cannot exhibit initial undershoot with positive input. Thus, they also cannot exhibit overshoot as the steady state is approached. Therefore, the step response is monotonic and the 1% settling time is simply the 100% response time ($n\tau_a$) minus the 1% response time. By expansion of Eq. (32), it is seen that the nth order impulse response, for $0 \leqslant t \leqslant n\tau_a/2$, is $1/\tau_a$ times the nth term in the (MacLaurin) expansion of $\exp(t/\tau_a)$. It is then easy to show that the 1% settling time is

$$t_{1\%\ \mathrm{settling}} = [n - (n!\times 10^{-2})^{1/n}]\tau_a. \quad (34)$$

The noise bandwidth is obtained by using the tabulated integral[30]

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin anx}{x}\,dx = \frac{\pi}{2}\left(1 - \frac{1}{2^{n-1}n!} \times \sum_{k=0}^{k<(n/2)(1-a)} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} (n - an - 2k)^n\right), \quad (35)$$

where a is real, $n \geqslant 1$, $[{n \atop k}] = n(n-1)...(n-k+1)/k!$, and $[{n \atop 0}] \equiv 1$. Letting $a \equiv 1/n$, and noting that the upper summation limit is 0 if $(1-1/n)n/2$ is zero, yields

$$B_n = \frac{1}{2\tau_a}\left(1 - \frac{1}{2^{n-1}n!} \sum_{k=0}^{k<(n-1)/2} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \times (n-1-2k)^n\right), \quad (36)$$

where $n = 1$ gives $B_n = 1/2\tau_a$, as expected. For m stages, n in Eqs. (35) and (36) is $2m - 1$.

CASCADED, IDEAL (FREQUENCY DOMAIN) LPFs: The signal and noise bandwidths of a cascade of ideal ("brickwall") LPFs are the single-stage bandwidth [Eq. (8)] while the phase is the sum of the phases, possibly zero. Since the ideal LPF has acausal impulse response and step response, it may be further idealized, for comparison purposes, by replacing the step response with a piecewise linear approximation, with slopes and domains given by[31]

$$\text{idealized step response slopes} = \begin{cases} 0 & \text{for } t<0 \\ 2Bt & \text{for } 0 \leqslant t \leqslant 1/2B \\ 0 & \text{for } 1/2B < t \end{cases} \quad (37)$$

where the nonzero slope is the maximum slope. By coincidence, the settling time of such an idealized filter is precisely equal to that of the gated or running integrator.

CASCADED, GAUSSIAN LPFs: The transfer function of a Gaussian LPF is

$$H(f) = H(0)e^{-(\ln 2/2)(f/B)^2}e^{-i\omega t_0}, \quad (38)$$

where B is the -3-dB bandwidth. Gaussian filters are ideal in the time domain[32] and exhibit no overshoot. Like the ideal LPF, they are not physically realizable. For n cascaded Gaussian filters, each with unity dc gain and equal signal bandwidths, the power transfer function is

$$|H(f)|^2 = e^{-n \ln 2(f/B)^2} = (e^{-\ln 2})^{n(f/B)^2}. \quad (39)$$

The signal bandwidth, in terms of B, is obtained by letting Eq. (39) = 1/2, giving

$$f_{-3\text{ dB}} = n^{-(1/2)}B, \quad (40)$$

where n is the number of cascaded filters. The noise bandwidth is given by substituting Eq. (39) into (10)

$$B_n = (1/2)(\pi/\ln 2)^{1/2}n^{-(1/2)}B, \quad (41)$$

so that $B_n f/_{-3\text{ dB}}$ is independent of n

$$\frac{B_n}{f_{-3\text{ dB}}} = \frac{1}{2}\left(\frac{\pi}{\ln 2}\right)^{1/2} \quad (42)$$

as in Eq. (27). This ratio is incorrectly given[33] as 1.34.

The step response of the Gaussian filter is acausal so the step response is further idealized with the nonzero slope given by the maximum slope[34]

$$\text{idealized step response slopes} = \begin{cases} 0 & \text{for } t<0 \\ \left(\frac{2\pi}{\ln 2}\right)^{1/2} Bt & \text{for } 0 \leqslant t \leqslant \frac{1}{B}\left(\frac{\ln 2}{2\pi}\right)^{1/2} \\ 0 & \text{for } \frac{1}{B}\left(\frac{\ln 2}{2\pi}\right)^{1/2} < t \end{cases}. \quad (43)$$

Since the cascaded Gaussian filter is also Gaussian, its rise-time-bandwidth product must be independent of n. Then Eq. (40) or (41) implies that the 1% settling time is $\sqrt{n}$ times greater than the single stage 1% settling time.

III. NUMERICAL RESULTS

A real, causal, low-pass filter has a voltage transfer function given as the ratio, a proper fraction, of two polynomials in the Laplace transform variable p

$$H(p) = k\frac{a_0 + a_1 p + ... + a_m p^m}{b_0 + b_1 p + ... + b_n p^n} \equiv k\frac{N(p)}{D(p)}, \quad (44)$$

where i = the square root of -1, ω is the angular frequency (rad/s), $s = i\omega$, $p = s\tau$, $N(p)$ is the numerator polynomial, and $D(p)$ is the denominator polynomial. Roots of $D(p)$ and $N(p)$ are called poles and zeros, respectively. The dc gain is ka_0/b_0. Factoring out a_m/b_n gives an alternate expression for $H(p)$

$$H(p) = K\frac{(a_0/a_m) + (a_1/a_m)p + ... + p^m}{(b_0/b_n) + (b_1/b_n)p + ... + p^n}, \quad (45)$$

where K is the value in Eq. (44) times a_m/b_n and is called the (numerator) scale factor. The expression for $H(p)$ in Eq. (45) may be factored to yield the following form:

$$H(p) = K\frac{(p+Z_1)(p+Z_2)...(p+Z_m)}{(p+P_1)(p+P_1)...(p+P_n)}. \quad (46)$$

A computationally convenient alternative form of Eq. (44) is the nested form:

$$H(p) = k\frac{a_0 + p(a_1 + p(... + pa_m)...)}{b_0 + p(b_1 + p(... + pb_n)...)}. \quad (47)$$

All of these expressions are in widespread use so some care is needed to avoid confusion.

A proper comparison of LPFs for averaging purposes must involve a renormalization of the poles, zeros (if any), and scale factors of the LPFs to be compared. This is quite easy, since these quantities, which completely determine the filter transfer function, are extensively tabulated for a variety of filters, and the normalization is usually to 1-rad/s signal bandwidth, 1-s time constant, or 1-rad signal bandwidth-time constant product. It is only necessary to convert the poles and zeros to rectangular form and calculate the filter noise bandwidth, then renormalize the poles, zeros, and scale factor to 1-Hz noise bandwidth. It should also be noted that poles and zeros may be given in a variety of nonrectangular forms such as undamped natural frequencies and damping ratios.[35]

In addition to the variety of explicit transfer function normalizations employed, it is common to implicitly normalize to $\tau = 1$ s, i.e., p and s are used interchangably. For some filters, e.g., Bessel filters, this may be the only normalization. In any event, it is best to calculate both $\omega_{-3\text{ dB}}\tau$ (the signal bandwidth-time constant product) and $\omega_n\tau$ (the noise bandwidth-time constant product) and then use the information so obtained to renormalize to 1-Hz noise bandwidth. In the present case, $\omega_{-3\text{ dB}}\tau$ and $\omega_n\tau$ were calculated using a short Hewlett–Packard HP-15C calculator program using the transfer function in nested form [Eq. (47)]. Figure 1 shows the transfer function magnitudes vs $\omega\tau$ for the five best LPFs and for the third-order Bessel LPF.

The calculator was used because it directly calculates with complex numbers and polynomials and has built-in solve and integrate functions which make the necessary calculations almost trivial. To check the accuracy of the numerical integrations, the numerical noise bandwidth-time constant products were first obtained and then the analytic expressions were derived (if possible). For the third-order LPFs, which comprise the majority of filters in this study, the analytic expression given by Pickup[10] was used to check the numerical results. In no case was the error greater than 0.1% if care was taken to properly specify the upper integration limit. For the all pole LPFs, the upper limit was selected so that the power transfer function was $\leqslant 10^{-4}$ for a specified integration precision of 10^{-3}. For the (2, 3) LPFs, the upper limit was selected so that the power transfer function was $< 10^{-8}$, i.e., -80 dB. In addition, the integration interval was subdivided and several integrals were summed to determine whether a single numerical integration was sufficiently accurate to be trustworthy. Such subdivision was found to be beneficial primarily for filters with zeros. It should also be noted that the analytic approach used by Pickup, based on spectral factorization tables[36], is essentially useless for high-order filters such as the tenth-order Bessel filter. Also, it would be difficult to handle transitional filters, such as the transitional Butterworth–Thompson filters, by the analytic approach. The Paynter filter, described as the "optimum" analog approach to running integration,[37] is the "mean," third-order, transitional Butterworth–Thompson filter.

Having obtained the transfer function normalized to 1-Hz noise bandwidth, the time-domain response to a unit step input is obtained by taking the inverse Laplace transform of $H(p)/s$, where the factor of $1/s$ is the Laplace transform of a unit step function. A simple BASIC program to perform the inverse transform numerically, on factored transfer functions [Eq. (46)], has been published.[38] From data generated in the inverse transform, which requires as much as 10 min for the tenth-order Bessel filter, it is possible to plot the unit step response and determine (by, e.g., linear interpolation) the settling time of the filter. Figure 2 shows time domain results for the five best LPFs and for the third-order Bessel LPF. Note that restricting the settling tolerance to, e.g., $\pm 0.5\%$ causes the Grimbleby and Bessel LPFs to be the best of the six responses shown. The other four LPFs, having been optimized for 1% settling, become inferior. It should also be observed that the specific noise bandwidth normalization used does not alter the relative performance of the LPFs, but this is not the case for the settling tolerance.

Table I gives the rectangular form poles, zeros, scale factors, normalizations, and references for the third-order filters evaluated numerically. Bessel filter poles and gains are taken from Weinberg[39] and $\omega_n/\omega_{-3\text{ dB}}$ ratios for $n = 2, 4, 6,$ and 8 are in agreement with graphical results.[40] The Paynter filter is from Beauchamp.[41] From the numerically derived $\omega_n\tau$ values and the $B_n \equiv 1$-Hz definition, τ is obtained. Note

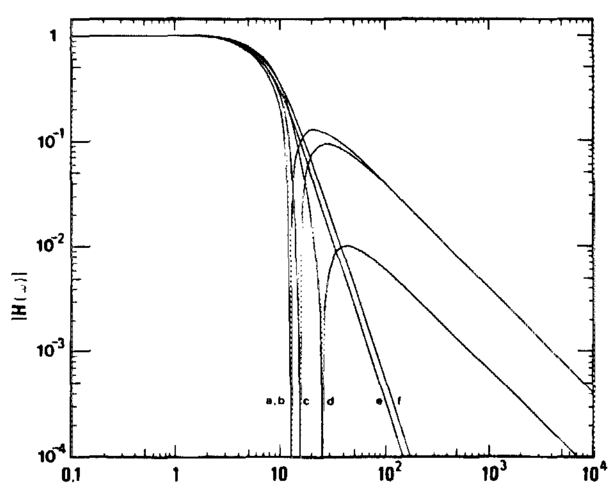

FIG. 1. Transfer function magnitudes vs $\omega\tau$ for the Pickup 1% (a), modified Sheingold (b), Grimbleby (c), Jess and Schüssler 32 (d), Jess and Schüssler 30 (e), and Bessel (f) low-pass filters.

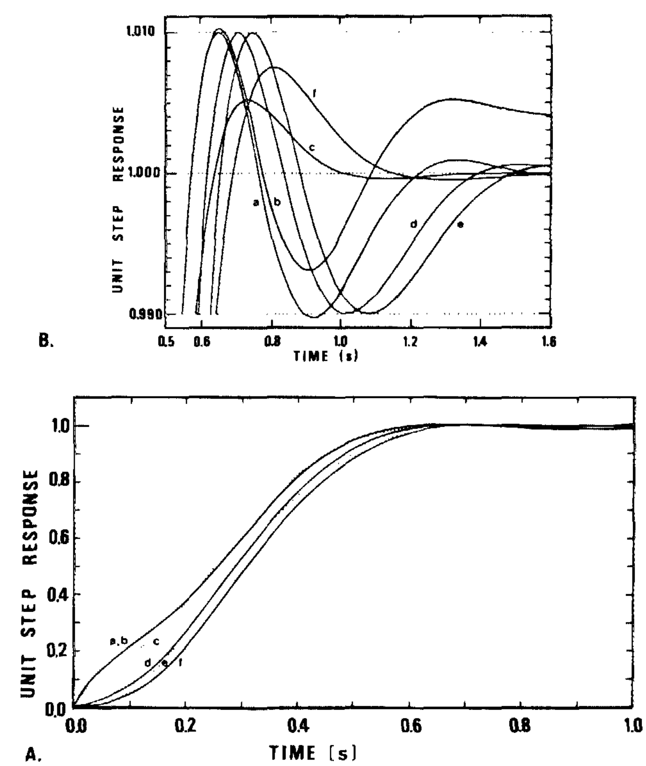

FIG. 2. (A) The unit step response of the 6 filters in Fig. 1. Notation as in Fig. 1. (B) The $\pm 1\%$ settling region of (A).

TABLE I. Third-order low-pass filter scale factors, poles, and zeros.

Name	Scale factor	Poles	Zeros	Norm	Reference
Gaussian approximant	4.196	1.5116, 1.375 $\pm$ 0.9411i		τ;$\omega_{-3\ \mathrm{dB}}$	[27, p.252]
Guillemin	2.00	1.0000, 1.0000 $\pm$ 1.0000i		τ	[8, p.141]
Bessel	15.00	2.3222, 1.8389 $\pm$ 1.7544i		τ	[39, p.500]
Least-squares delay	1.962	1.043, 0.853 $\pm$ 1.074i		τ;$\omega_{-3\ \mathrm{dB}}$	[9, p.129]
Jess & Schüssler 30	0.00988	0.1617, 0.1316 $\pm$ 0.2092i		τ;$\omega_{-40\ \mathrm{dB}}$	[27, p.299]
Jess & Schüssler 32	0.0297	0.2575, 0.2041 $\pm$ 0.3335i	0 $\pm$ 1.1521i	τ;$\omega_{-40\ \mathrm{dB}}$	[27, p.299]
Grimbleby	4.000	9.2887, 7.3556 $\pm$ 7.0175i	0 $\pm$ 15.492i	τ;B_n	[10, p.604]
Pickup 1%	3.971	6.5122, 4.7134 $\pm$ 8.6908i	0 $\pm$ 12.661i	τ;B_n	[10, p.604]
Montonic-L	0.577	0.6203, 0.3452 $\pm$ 0.9008i		τ;$\omega_{-3\ \mathrm{dB}}$	[27, p. 229]
Parabolic $b = 1.25$	1.239	0.7909, 0.6259 $\pm$ 1.084i		τ;$\omega_{-3\ \mathrm{dB}}$	[27, p.260]
Parabolic $b = 3$	1.394	0.7771, 0.6696 $\pm$ 1.160i		τ;$\omega_{-3\ \mathrm{dB}}$	[27, p.260]
Catenary	1.177	0.8027, 0.6056 $\pm$ 1.049i		τ;$\omega_{-3\ \mathrm{dB}}$	[27, p.264]
Elliptic contour	2.183	1.2044, 0.8975 $\pm$ 1.0034i		τ;$\omega_{-3\ \mathrm{dB}}$	[27, p.268]
Linear phase minimax	11.64	1.8136, 1.4394 $\pm$ 2.0846i		τ	[27, p.306]
Butterworth	1.000	1.0000, 0.5000 $\pm$ 0.8660i		τ;$\omega_{-3\ \mathrm{dB}}\tau$	[39, p.495]
Paynter	1.000	0.7677, 0.5487 $\pm$ 1.001i		$\omega_{-3\ \mathrm{dB}}\tau$	[41, p.141]
2,3-Padé approximant	3.000	3.6378, 2.6811 $\pm$ 3.0504i	$-$ 4.000 $\pm$ 2.000i	τ	[27, p.311]

that the gain must be divided by τ^{n-m}, where n is the denominator order and m is the numerator order.

Table II shows the 1-Hz noise bandwidth normalized poles, zeros, and scale factors of the filters in Table I. In addition, the 1% settling times are also shown, calculated as described above. Evidently, filters below the modified Sheingold have mediocre time domain performance, i.e., substantial overshoot by the 1% settling criterion, and are thus not usable as averaging filters. In particular, the Paynter filter, with 2.9% overshoot, offers poor performance. However, if the required measurement resolution is such that the 1% settling tolerance must be replaced by another settling tolerance, then the relative ordering of the filters in Table II is changed. Since the step response data sets have already been generated, it is easy to reexamine them for the settling times corresponding to alternate settling tolerances. It is also easy to regenerate the unit step responses from the scale factors, poles, and zeros (if any), for the filters in Table II. We leave this to the interested reader.

Even if the settling time criterion was replaced by a simple response time criterion, such as 99% response, these filters would be unable to respond in less than 0.6 s. The only unsuitable filter which can be slightly modified to achieve 1% settling in under 0.6 s is the Sheingold filter[42] which has 0.5% overshoot and 1.1% undershoot. A slight reduction (0.88%) in the real part of the complex pole and renormalization to 1-Hz noise bandwidth, gives 1.015% overshoot, 0.7% undershoot, and 1.004 dc gain—slightly violating the first LPF comparison condition. As may be seen from the data in Table II, the modified Sheingold and Pickup 1% filters have quite similar transfer functions and performance with the Pickup filter having 1% overshoot, 1.015% undershoot, and faster ultimate settling. These filters are easily and inexpensively constructed in nontunable form.

TABLE II. Third-order low-pass filter gains, poles, zeros, and 1% settling times for 1.000 $\pm$ 0.0005 Hz noise bandwidth normalization.

Name	Scale factor	Poles $-\alpha, -\beta \pm i\gamma$	Zeros $-z_1 \pm iz_2$	$t_{1\%}$ (s)	B_n' (Hz)
Gaussian approximant	764.8	8.571, 7.795 $\pm$ 5.366i	—	0.703	1.462
Guillemin	592.6	6.667, 6.667 $\pm$ 6.667i	—	0.699	1.449
Bessel	555.6	7.741, 6.130 $\pm$ 5.848i	—	0.645	1.413
Least-squares delay	425.4	6.266, 5.125 $\pm$ 6.452i	—	0.631	1.393
Jess & Schüssler 30	340.9	5.265, 4.285 $\pm$ 6.811i	—	0.628	1.395
Jess & Schüssler 32	0.6519	5.658, 4.485 $\pm$ 7.328i	0 $\pm$ 25.32i	0.594	1.380
Grimbleby	4.000	9.289, 7.356 $\pm$ 7.018i	0 $\pm$ 15.49i	0.588	1.499
Pickup 1%	3.971	6.512, 4.713 $\pm$ 8.691i	0 $\pm$ 12.66i	0.550	1.535
Modified Sheingold	3.961	6.338, 4.712 $\pm$ 8.823i	0 $\pm$ 12.68i	0.550	1.532
Monotonic-L	148.8	3.947, 2.197 $\pm$ 5.733i	—	1.59	1.213
Parabolic $b = 1.25$	309.1	4.979, 3.940 $\pm$ 6.824i	—	1.22	1.379
Parabolic $b = 3$	339.3	4.852, 4.181 $\pm$ 7.242i	—	1.17	1.421
Catenary	294.4	5.057, 3.815 $\pm$ 6.607i	—	1.24	1.369
Elliptic contour	459.6	7.165, 5.339 $\pm$ 5.970i	—	0.900	1.384
Linear phase minimax	362.9	5.708, 4.530 $\pm$ 6.561i	—	0.832	1.382
Butterworth	216	6.000, 3.000 $\pm$ 5.196i	—	1.57	1.219
Paynter	266.5	4.940, 3.531 $\pm$ 6.440i	—	1.31	1.354
2,3-Padé approximant	2.933	3.557, 2.622 $\pm$ 2.983i	$-$ 3.911 $\pm$ 1.955i	2.16	1.517

Examples of active RC realization of the Pickup 0.1% and Sheingold LPFs are given in the appropriate references. The authors are not sufficiently versed in filter circuit design praxis to suggest optimum circuitry for the Pickup 1% and modified Sheingold LPFs. Our goal is to point out the intrinsically significant design goal.

Figure 3 shows the 1% settling times of the synchronous, integrator, ideal, and Gaussian LPFs. Also shown are the Bessel LPF results obtained numerically. Clearly, the only advantage of the synchronous filter is its simplicity (i.e., cheapness). Note that common filters such as elliptic and Chebyshev filters are not shown in Fig. 3 because their time domain behavior is atrocious.[43] Other filters omitted for the same reason include ultraspherical, inverse Chebyshev, modified inverse Chebyshev, and Paynter LPFs.[44] Filters for which too little data is available to make a comparison include equiripple delay, prolate, least-squares magnitude, Halpern, and Aronhime and Budak LPFs.[44] Excepting the Jess and Schüssler filters, none of the numerically constrained LPFs given by Lindquist had acceptable step response overshoot.[45]

Figure 4 shows the settling time performance of LPFs within the dotted box in Fig. 3. Note that best performance is achieved with filters having zeros and that the Jess and Schüssler LPFs, though optimized for minimum step response rise-time stop band-edge product, are inferior to the Pickup and modified Sheingold LPFs for averaging purposes. The reason is simple: an arbitrary rise-time-bandwidth figure of merit was "optimized"[46] rather than the settling time-noise bandwidth product.

We must emphasize again that the figure of merit, of a low-pass filter intended for averaging purposes, is the settling time-noise bandwidth product. The settling tolerance is determined by the required measurement resolution. With the reasonable, but arbitrary, assumption of 1% measurement resolution, the figure of merit is the 1% settling time-noise bandwidth product and Figs. 3 and 4 give relative filter performances. For measurement resolutions other than 1%, the relative performances of the various LPFs is altered and Figs. 3 and 4 do not apply. As mentioned earlier, the third-order Bessel LPF is a good overall choice if slowly varying ac signals must also be acquired.

It is important to realize that filters of order higher than three do not provide superior performance. In part, this is due to the white-noise assumption because the faster roll off of higher order filters offers no advantage when interferences are absent. The noise bandwidth normalization completely suffices under these circumstances.

If, however, the noise power spectrum is nonwhite, the noise bandwidth concept must be modified. For example, Mossotti[47] defines the "equivalent number of degrees of freedom," denoted by η_e, for the nonwhite noise case. This provides a measure of the number of independent measurements that can be made in a given measurement period. Expressing η_e in Hz, rather than units of $1/T$ (the Nyquist cointerval), then gives[47]

$$\frac{\eta_e}{2T} = B_n' = \frac{(\int_0^\infty P(f)df)^2}{\int_0^\infty P^2(f)df} = \frac{(\int_0^\infty |H(f)|^2 df)^2}{\int_0^\infty |H(f)|^4 df}, \tag{48}$$

where B_n' is the effective bandwidth and is given in Table II for comparison. Note that B_n' is identical to a proposed new effective noise bandwidth,[48,49] a fact which derives from the

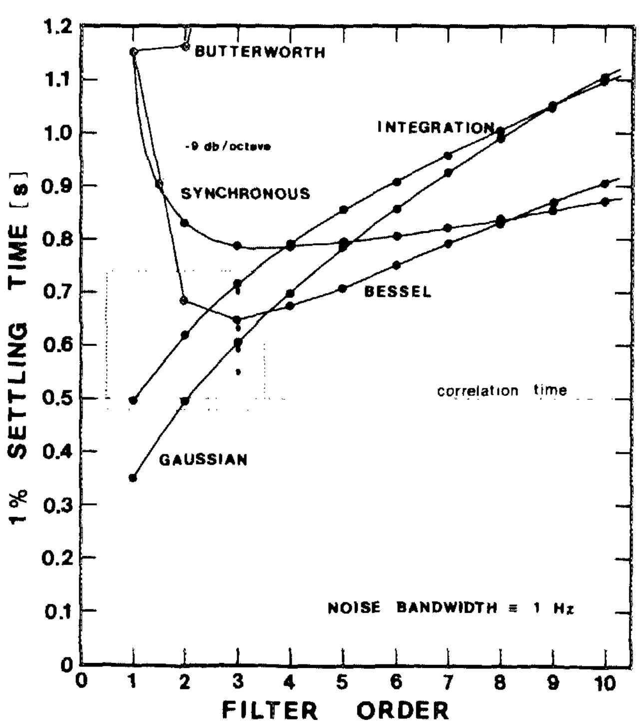

FIG. 3. Settling time (1%) vs filter order at 1-Hz noise bandwidth for several types of low-pass filter.

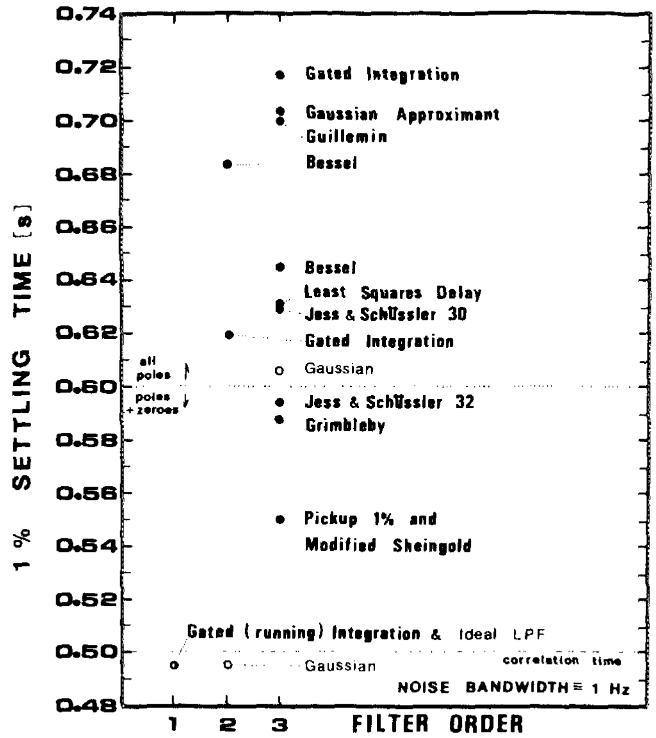

FIG. 4. Settling time (1%) vs filter order at 1-Hz noise bandwidth for the filters in the dotted outline in Fig. 3.

chi-squared distribution of output noise powers assumed in each case.[50]

IV. CONCLUSIONS

As may be seen from Figs. 3 and 4, the 1% settling time of the running integrator is 0.495 s since exact settling occurs in 0.5 s. The Pickup 1% and modified Sheingold LPFs settle in 0.55 s which is only about 11% higher. Furthermore, the Pickup 0.1% LPF settles to 1% in 0.60 s and to 0.1% in 0.67 s.[10] The *RC* LPF settles to 0.1% in 1.73 s. Thus, there is little reason to use *RC* LPFs and, in particular, the first-order *RC* LPF, unless such use is practically unavoidable, as in lock-in amplifiers used in feedback systems.[51] Otherwise, the use of synchronous LPFs in instruments such as lock-in amplifiers and boxcar integrators/averagers should be avoided. At the very least, tunable third- or fourth-order Bessel filters, which are easily implemented with single integrated circuit "chips" such as the National Semiconductor MF-10 switched capacitor filter,[52] should be used. Practically, this means that the "time constant" on the instrument should be minimized and the (slightly filtered) output should then be processed by the "outboard" LPF.

Tunable LPF designs as good as or better than the Pickup 1% and modified Sheingold LPFs would be most useful because it is the noise reduction properties of the output LPF and the allotted measurement time which ultimately determines the detection limit (detectivity) of a technique. The detection limit is an important figure of merit for techniques and systems because it is an easily measured, intrinsic quantity that determines just how "sensitive" a technique is. Lowered detection limits are useful because they translate into reduced measurement times or results of higher quality, i.e., S/N, in fixed measurement time. In practice, the detection limit is just a small "protection factor," typically 1, 2, 3, or 6, times an appropriate "noise equivalent measure." Examples of noise equivalent measures include noise equivalent power (NEP), noise equivalent pressure, and noise equivalent concentration. Note that the noise equivalent measure is just that measure for which $S/N = 1$. Thus, for a white-noise-dominated situation, the noise may be indefinitely reduced at the expense of measurement time and comparisons of two, quite similar, detection procedures[53] may well be meaningless even if noise equivalent measures are used.

A necessary condition for a valid comparison of similar systems is that the systems are properly designed: the dominant noise is white, or whitened, by design and is known to be white by valid, experimental verification. More often than not, this condition is not strictly met; when it is, it is possible to compare the detection limits of the systems if both the protection factors and the noise bandwidths are equalized. In this regard, the comparison of infrared detector D^* values[54] is the appropriate parallel. Unfortunately, the detection limit definitions in common use are incomplete, like the infrared detector detectivity value, and most comparisons of systems are flawed for nonsubtle reasons. It is hoped that the results presented here will help correct the problem.

ACKNOWLEDGMENTS

The authors wish to gratefully acknowledge support from NIH (5R01 GM11373-22) and DOE (DE-AS05-78R06022). They also wish to thank the reviewers, particularly the first reviewer, for their considerable and thoughtful efforts in improving this paper.

[a] Author to whom correspondence is to be addressed.

[1] D. P. Blair and P. H. Sydenham, J. Phys. E **8**, 621 (1975).
[2] R. G. Brown, *Introduction to Random Signal Analysis and Kalman Filtering* (Wiley, New York 1983), p. 81.
[3] L. A. Wainstein and V. D. Zubakov, *Extraction of Signals from Noise*, translated by R. A. Silverman (Dover, New York, 1970), p. 14.
[4] A. Papoulis, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill, New York, 1965), p. 405.
[5] A. Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977), p. 330.
[6] S. E. Tavares, IEEE Trans. Instrum. Meas. **IM-15**, 33 (1966).
[7] W. B. Davenport, Jr. and W. L. Root, *An Introduction to the Theory of Random Signals and Noise* (McGraw-Hill, New York, 1958), p. 175.
[8] A. F. Arbel, *Analog Signal Processing and Instrumentation* (Cambridge Univ., Cambridge, 1980), p. 334.
[9] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), p. 315.
[10] C. P. Pickup, J. Phys. E **14**, 602 (1981).
[11] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), pp. 53–56.
[12] K. R. Betty and G. Horlick, Anal. Chem. **48**, 2248 (1976).
[13] M. R. Thompson and R. E. Dessy, Anal. Chem. **56**, 583 (1984).
[14] Oriel Corp. Catalog, Oriel Instrumentation for Spectroscopy & Detection, UV-VIS-IR, 11/83, pp. 72–73.
[15] A. B. Carlson, *Communication Systems: An Introduction to Signals and Noise in Electrical Communication*, 2nd ed., (McGraw-Hill, New York, 1975), p. 139.
[16] A. F. Arbel, *Analog Signal Processing and Instrumentation* (Cambridge Univ., Cambridge, 1980), pp. 141–144.
[17] W. D. Gregg, *Analog and Digital Communication* (Wiley, New York, 1977), pp. 468–477.
[18] K. Neelakantan and S. Dattagupta, Pramana **11**, 295 (1978).
[19] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), p. 149.
[20] C. D. McGillem and G. R. Cooper, *Continuous and Discrete Signal and System Analysis* (Holt, Rinehart and Winston, New York 1974), pp. 120–121.
[21] M. Schwartz, *Information Transmission, Modulation, and Noise*, 3rd ed. (McGraw-Hill, New York 1980), pp. 117–120.
[22] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), pp. 139–146.
[23] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), pp. 142–144.
[24] I. S. Gradshteyn and I. M. Ryzhik, in *Table of Integrals, Series, and Products Corrected and Enlarged Edition*, edited by A. Jeffrey (Academic, New York, 1980), p. 294, 3.249(1).
[25] G. Arfken, *Mathematical Methods for Physicists*, 2nd ed. (Academic, New York, 1970), pp. 457–458.
[26] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), p. 149.
[27] C. S. Lindquist, *Active Network Design with Signal Filtering Applications* (Steward, Long Beach, CA, 1970), p. 270.
[28] J. H. Moore, C. C. Davis, and M. A. Coplan, *Building Scientific Apparatus* (Addison–Wesley, New York, 1983), p. 436.
[29] Ref. 24, p. 457, 3.836(2).
[30] Ref. 24, p. 458, 3.836(4).
[31] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), pp. 83, 84.
[32] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), p. 87.
[33] F. N. H. Robinson, *Noise and Fluctuations in Electronic Devices and Circuits* (Clarendon, Oxford, 1974), p. 119.
[34] H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiely, New York, 1976), pp. 86, 87.

[35]J. J. DiStefano, III, A. R. Stubberud, and I. J. Williams, *Feedback and Control Systems,* Schaum's Outline Series (McGraw-Hill, New York, 1967), pp. 39, 40.
[36]Ref. 2, p. 12c and original references therein.
[37]T. C. O'Haver, Anal. Chem. **50**, 676 (1978).
[38]B. Finlay, Byte, **4**, 144 (1979).
[39]L. Weinberg, *Network Analysis and Synthesis,* (McGraw-Hill, New York, 1962), p. 500.
[40]H. J. Blinchikoff and A. I. Zverev, *Filtering in the Time and Frequency Domains* (Wiley, New York, 1976), p. 148.
[41]K. G. Beauchamp, *Signal Processing Using Analog and Digital Techniques* (Allen & Unwin, London, 1977), p. 144.
[42]D. H. Sheingold, editor, *Nonlinear Circuits Handbook* (Analog Devices, Norwood, MA 1974), pp. 105, 106.
[43]Ref. 27, pp. 241 and 224, respectively.
[44]Ref. 27, Chap. 4.
[45]Ref. 27, Chap. 5.
[46]Ref. 27, pp. 297–299.
[47]V. G. Mossotti, in *Treatise on Analytical Chemistry, Part I, Theory and Practice,* V. 4, 2nd ed. edited by P. J. Elving and V. G. Mossotti (Wiley, New York), pp. 45–47.
[48]P. Kittel, Rev. Sci. Instrum. **48**, 1214 (1977).
[49]M. J. Roberts and T. V. Blalock, Rev. Sci. Instrum. **56**, 326 (1985).
[50]A. D. Whalen, Detection of Signals in Noise (Academic, New York, 1971), pp. 109–112.
[51]A. Lock-In Primer, EG&G Princeton Applied Research, Princeton, NJ.
[52]National Semiconductor MF10 Universal Monolithic Dual Switched Capacitor Filter, 4/82 data sheet.
[53]P. W. J. M. Boumans, Spectrochimica Acta **33B**, 625 (1978).
[54]R. H. Kingston, *Optical and Infrared Radiation* (Springer Berlin, 1978) pp. 16, 17.