3

SIMPLE APPLICATIONS

With the help of the concept of averages of random variables, we can already understand the limiting accuracy of some simple measurements due to the inherent noise in the measuring system.

As a first case we consider noise in electrical circuits (Section 3.1a) and show that it becomes significant when the signal to be processed lies within the μV range. As a second case we consider noise in sensitive galvanometers (Section 3.1b) and show that it limits the accuracy of current measurements when the currents are of the order of 1 pA.

Much more sensitive measurements can be performed when counting techniques can be introduced (Section 3.2). For example, electron currents in a vacuum can be multiplied by an electron multiplier so that individual electrons can be counted; very small currents can be measured in this manner (Section 3.2a). Or a stream of photons can be made to impinge upon the photocathode of a photomultiplier and the individual photons can be counted; very small radiant powers can be determined in this manner (Section 3.2b).

3.1 NOISE IN ELECTRICAL MEASUREMENTS

When small particles are suspended in a liquid, they execute a random zigzag motion, called *Brownian motion*, named after its discoverer Robert Brown (1827). The problem was extensively studied during 1890–1910, and Einstein developed its theory in 1905. Einstein suggested that the mean kinetic energy per degree of freedom of the particle should be given by statistical mechanics as

$$\tfrac{1}{2}M\,\overline{v_x^2} = \tfrac{1}{2}kT \tag{3.1}$$

where M is the mass of the particle, v_x its instantaneous velocity component in the X direction, $\overline{v_x^2}$ its mean-square value, k the Boltzmann constant, and T the absolute temperature. However, what is actually *observed* under the microscope is not the instantaneous velocity v_x, but the displacement Δx in the X direction during the time interval t. Einstein showed that

$$\overline{\Delta x^2} = 2Dt \tag{3.2}$$

where $\overline{\Delta x^2}$ is the mean-square value of Δx and D is the diffusion constant of the particle. This equation was later verified experimentally.

Einstein quickly realized that many physical systems would show Brownian motion. For example, thermal noise is nothing but "Brownian motion of electricity" in electrical circuits. We discuss here the case of the input circuit of a wideband amplifier and the case of a sensitive galvanometer.

3.1a Noise in Electrical Circuits

As an example of an electrical circuit we consider the RC circuit of Fig. 3.1. As a result of the random thermal agitation of the electrons in the resistor R the capacitor C will be charged and discharged at random. In analogy with (3.1) the average energy stored in the capacitor will be

$$\tfrac{1}{2}C\overline{v^2} = \tfrac{1}{2}kT, \quad \text{or} \quad \overline{v^2} = \frac{kT}{C} \tag{3.3}$$

The value of $(\overline{v^2})^{1/2}$ is in the μV range. Take $C = 30$ pF, $T = 300°$K, $k = 1.38 \times 10^{-23}$ joule deg^{-1}, then $\overline{v^2} = (1.38 \times 10^{-10}\ \text{V}^2)$ or $(\overline{v^2})^{1/2} \simeq 12\ \mu$V. If we must process electronic signals with the help of wideband amplifiers, we thus run into noise problems in the input circuit if the input-signal level is in the μV range.

It should be noted that the resistor R does not enter into the final result; it only determines how wide the passband of a wideband amplifier processing the noise signal must be in order to measure the full voltage developed across the capacitor C.

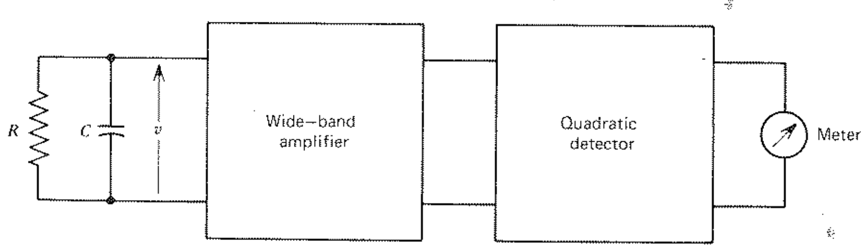


Figure 3.1. The thermal noise voltage v across an RC circuit is amplified by a wide-band amplifier, passed through a quadratic detector, and measured.

The measuring arrangement of $\overline{v^2}$ is shown in Fig. 3.1. The electrical circuit is connected to a wideband amplifier that amplifies the voltage v by a factor $g \gg 1$. The output voltage is fed into a quadratic detector that gives an output $g^2\overline{v^2}$ and the output of the quadratic detector is fed into a meter M, so that the deflection of the meter is proportional to $\overline{v^2}$.

Quite often the problem is different. For example, one wants to measure a small d.c. voltage v_0 or a small single frequency a.c. signal $v_0 \cos\omega t$. The noise is then a disturbing factor and one wants to make it as small as possible. It is then important to filter most of the noise out by means of a narrow-band filter (a lowpass filter in the d.c. case and a bandpass filter of center frequency ω for the a.c. case). To evaluate the filter output one must make a Fourier analysis of the noise; that problem is discussed in Chapter 5.

We finally prove the equipartition theorem (3.3). If a system has a temperature T, the probability that it has an energy E is proportional to $\exp(-E/kT)$; the factor $\exp(-E/kT)$ is called the *Boltzmann factor*. In the RC circuit the energy stored in the capacitor C is $\frac{1}{2}Cv^2$, where v is the voltage across C. The probability dP of finding a voltage between v and $(v + dv)$ is therefore

$$dP = C_0 \exp\left(-\frac{\frac{1}{2}Cv^2}{kT}\right)dv \tag{3.4}$$

where the constant C_0 must be so chosen that

$$\int_{-\infty}^{\infty} dP = 1 \quad \text{or} \quad C_0\int_{-\infty}^{\infty} \exp\left(-\frac{\frac{1}{2}Cv^2}{kT}\right)dv$$

$$= C_0\left(\frac{2kT}{C}\right)^{1/2}\int_{-\infty}^{\infty}\exp(-x^2)\,dx$$

$$= \left(\frac{2kT}{C}\right)^{1/2}\pi^{1/2}C_0 = 1, \quad \text{or} \quad C_0 = \left(\frac{C}{2\pi kT}\right)^{1/2} \tag{3.4a}$$

Here $x^2 = (Cv^2/2kT)$. Therefore

$$\overline{v^2} = C_0\int_{-\infty}^{\infty} v^2 \exp\left(-\frac{Cv^2}{2kT}\right)dv = C_0\left(\frac{2kT}{C}\right)^{3/2}\int_{-\infty}^{\infty} x^2\exp(-x^2)\,dx$$

$$= C_0\left(\frac{2kT}{C}\right)^{3/2}\cdot\frac{\pi^{1/2}}{2} = \frac{kT}{C} \tag{3.4b}$$

as is found by substituting for C_0.

3.1b Noise in Galvanometers

Another example of Brownian motion is the random rotation of a galvanometer coil around its axis of suspension. Let us consider a galvanometer with electromagnetic damping, where the deflection of the coil is described by the differential equation

$$\Theta\frac{d^2\varphi}{dt^2}+n_0^2\frac{A^2B^2}{r}\frac{d\varphi}{dt}+D\varphi=n_0ABi(t) \tag{3.5}$$

Here Θ is the moment of inertia of the coil, A its area, n_0 its number of turns, B the magnetic flux density, $-D\varphi$ the restoring torque that tends to bring the coil back to its equilibrium position, and $i(t)$ the current flowing through the coil, giving rise to a driving torque $n_0ABi(t)$ acting on the coil. Finally r is the resistance of the loop circuit consisting of the coil resistance plus any external resistor.

The term $(nA^2B^2/r)(d\varphi/dt)$ represents the damping torque acting on the coil. To understand this term, we observe that $(d\varphi/dt)$ is the angular velocity of the coil; hence $-n_0AB(d\varphi/dt)$ is the induced e.m.f. caused by the coil motion and $-(n_0AB/r)(d\varphi/dt)$ is the current flowing through the circuit. Multiplying this expression by n_0AB to obtain the torque, we find for the damping torque $-(n_0^2A^2B^2/r)(d\varphi/dt)$. This torque occurs in the right-hand side of (3.5); by shifting it to the left-hand side we obtain (3.5) in its final form.

If the current $i(t)$ flowing through the coil is a direct current I, the steady-state solution of (3.5) is

$$\varphi=\frac{n_0AB}{D}I \tag{3.6}$$

The constant (n_0AB/D) is called the *technical sensitivity* of the galvanometer; it can be made large by making D small. However, if one does this, one runs into noise problems. For according to the equipartition theorem the average potential energy stored in the coil is

$$\tfrac{1}{2}D\overline{\varphi^2}=\tfrac{1}{2}kT, \qquad \text{or} \qquad \overline{\varphi^2}=\frac{kT}{D} \tag{3.7}$$

A small value of D thus corresponds to a large root mean square (r.m.s.) deflection $(\overline{\varphi^2})^{1/2}$ of the coil. The proof of (3.7) is similar to the earlier proof of (3.3).

Unfortunately, the constant D is not so easily measured; we should therefore express $\overline{\varphi^2}$ in more accessible parameters. Moreover, we are not so much interested in $(\overline{\varphi^2})^{1/2}$ itself, but more in the minimum current I_{min} that corresponds to it.

Galvanometers are practically always used under the condition of critical damping, since this gives the best and fastest response. To explain what this means we investigate the response of the galvanometer if $i(t)$ is a step function $au(t)$, where a is a small current step. To that end we introduce three parameters: (a) the resonant frequency ω_0, (b) the damping constant n, and (c) the time constant τ_0 by the definitions

$$\omega_0=\left(\frac{D}{\Theta}\right)^{\frac{1}{2}}; \qquad \frac{n_0^2A^2B^2}{r}=2n\omega_0\Theta, \qquad \tau_0=\frac{2\pi}{\omega_0} \tag{3.8}$$

Dividing (2.5) by Θ and making the substitutions we obtain

$$\frac{d^2\varphi}{dt^2}+2n\omega_0\frac{d\varphi}{dt}+\omega_0^2\varphi=\frac{n_0AB}{\Theta}au(t) \tag{3.9}$$

A standard mathematical analysis now shows that the solution of (3.9) is damped periodic for $n<1$ and aperiodic for $n>1$. The fastest and best response occurs for $n=1$; the galvanometer is then operating at the limit of aperiodicity and is said to be *critically damped*.

We now introduce a limiting d.c. current I_{min} by requiring that I_{min} gives a deflection equal to $(\overline{\varphi^2})^{1/2}$. Hence

$$I_{min}\frac{n_0AB}{D}=\left(\overline{\varphi^2}\right)^{1/2} \tag{3.10}$$

or

$$I_{min}=\left[\frac{D^2}{n_0^2A^2B^2}\cdot\frac{kT}{D}\right]^{1/2}=\left(\frac{DkT}{2n\omega_0\Theta r}\right)^{1/2}=\left(\frac{\pi kT}{nr\tau_0}\right)^{1/2} \tag{3.10a}$$

These quantities are easily measured.

Taking $r=1000\ \Omega$, $n=1$, $\tau_0=2$ s, $T=300°$ K yields $I_{min}=2.5\times10^{-12}$ A, so that galvanometer noise becomes important when measuring currents in the pA range.

3.2 MEASUREMENTS BY COUNTING TECHNIQUES

Suppose certain events, such as emission of electrons or emission of photons, occur at random instants at the average rate λ. Such events are called *Poisson* events. The question is now how to measure λ and how to determine the accuracy of the measurement.

If we observe during a time interval τ and N events are observed, then the average value of N is

$$\overline{N} = \lambda\tau, \qquad \text{or} \quad \lambda = \frac{\overline{N}}{\tau} \tag{3.11}$$

If the observation is repeated many times, the observed value of N will fluctuate around $\overline{N}$. Introducing the fluctuation $\Delta N = (N - \overline{N})$, it will be shown in Chapter 4 that for such events

$$\overline{\Delta N^2} = \text{var}\, N = \overline{N} = \lambda\tau \tag{3.12}$$

The inaccuracy of a single measurement is therefore $\Delta\lambda = (\Delta N/\tau)$ and its mean square value is

$$\overline{\Delta\lambda^2} = \overline{\left(\frac{\Delta N}{\tau}\right)^2} = \frac{\overline{\Delta N^2}}{\tau^2} = \frac{\lambda}{\tau} \tag{3.12a}$$

The r.m.s. inaccuracy of a single measurement thus varies as $1/\tau^{1/2}$. The larger one makes τ, the more accurate the measurement of λ becomes.

We now introduce the limiting accuracy of the measurement by asking what is the smallest value of λ that can be measured. We define it as the value of λ such that

$$\left[\overline{(\Delta\lambda)^2}\right]^{1/2} = \lambda, \qquad \text{or} \quad \left(\frac{\lambda}{\tau}\right)^{1/2} = \lambda, \qquad \text{or} \quad \lambda\tau = \overline{N} = 1 \tag{3.12b}$$

The limiting accuracy is thus obtained if on the average one count is made in τ seconds.

As an example consider the counting of electrons. Let the electrons be counted during 5 s. If, on the average, one electron is counted per interval, then the average current is

$$\overline{I} = \frac{e}{\tau} = \frac{1.6 \times 10^{-19}}{5} = 3.2 \times 10^{-20}\ \text{A}$$

This is many orders of magnitude better than the measurement of current with the help of galvanometers, indicating the power of the counting technique.

3.2a Current Measurement by Counting Techniques

Current measurements by counting techniques can be achieved if we have free electrons moving in a vacuum. We can then accelerate them and let them impinge upon the first dynode of a secondary emission multiplier

with m dynodes (Fig. 3.2). Let each dynode give δ secondary electrons per incident primary; then the output per incident primary is δ^m electrons. This can be made so large that individual pulses can be counted.

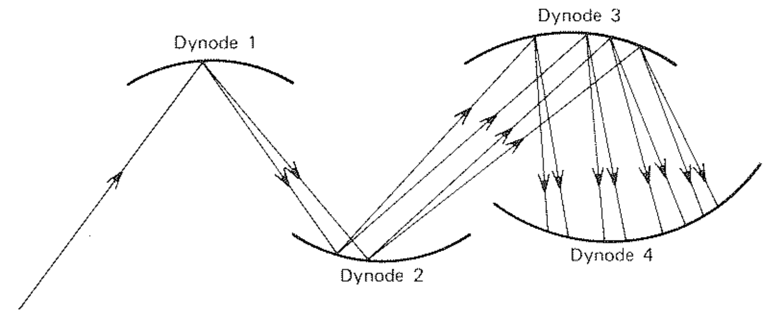


Figure 3.2. Electron multiplication in a secondary emission multiplier. An electron beam is incident upon a first dynode, multiplied, accelerated toward a second dynode, multiplied, acceletated toward a third dynode, multiplied, and so on.

There are two difficulties with this method, as the first dynode: (a) misses an occasional count and (b) emits electrons by thermionic emission. The first difficulty means that the first dynode has a probability p somewhat smaller than unity of producing a pulse when a primary electron strikes it. If the incoming electrons arrive independently and at random, then the pulses are also produced independently and at random. Hence if N electrons come in and n pulses are produced, then

$$\overline{n} = \overline{N}\,p; \qquad \text{var}\, n = \overline{\Delta n^2} = \overline{n} = \overline{N}\,p \tag{3.13}$$

If the observation time is τ, then the average rate of arrival is $\lambda = (\overline{N}/\tau)$ and the average rate of produced pulses is $(\overline{n}/\tau) = p\lambda$. Therefore,

$$\overline{\left(\frac{\Delta n}{\tau}\right)^2} = \frac{\overline{n}}{\tau^2} = \frac{\overline{N}p}{\tau^2} = \frac{p\lambda}{\tau} \tag{3.13a}$$

The limiting accuracy is now obtained if

$$\left(\frac{p\lambda}{\tau}\right)^{1/2} = p\lambda \qquad \text{or} \quad p\lambda\tau = 1 \qquad \text{or} \quad \overline{N} = \frac{1}{p} \tag{3.13b}$$

In secondary emission multipliers giving a multiplication δ per stage, p is approximately given by (see Chapter 4)

$$p = 1 - \exp(-\delta) \tag{3.14}$$

so that the effect is insignificant for $\delta > 4$, whereas it is quite important for smaller δ. In addition, some of the primary electrons are reflected by the dynode and do not produce secondary electrons; this is not taken into account by (3.14).

The second difficulty is best obviated by cooling the dynodes. The emission current I follows Richardson's law

$$I = CST^2 \exp\left(-\frac{e\Phi}{kT}\right) \tag{3.15}$$

where $C = 120\ \text{A/cm}^2$, S is the dynode area, Φ is the dynode work function in eV, and T is the dynode temperature. Hence by cooling the dynodes the exponential factor can be reduced to a very small value.

Secondary emission-multiplier techniques find extensive use in physics and engineering for measuring very small electron or ion currents in a vacuum.

3.2b Radiation Measurements by Counting Techniques

The same considerations apply to counting photons. If the photons come from a laser, the emission of photons occurs independently and at random, namely $\operatorname{var} N = \overline{N}$, where N is the number of emitted photons during a given time interval τ.

In carrying out the counting of the photons one must now use a photomultiplier. It consists of a photocathode emitting photoelectrons that are focused onto the first dynode of a secondary emission multiplier that multiplies the incident electrons and produces sufficiently large electrical pulses that can be counted.

The difficulty with this technique is that the photocathode has generally a low quantum yield η, defined as the average number of photoelectrons emitted per incident quantum. That is the price one has to pay for using photon-counting techniques. The theory of the previous section applies of course, provided that p is replaced by η. This means that the minimum number of photons that can be counted during τ seconds is

$$\overline{N} = \frac{1}{\eta} \tag{3.16}$$

Example: $\eta = 0.05$, photon energy $V_{ph} = 2.0$ eV; $\tau = 5$ s.

In this case $\overline{N} = (1/\eta) = 20$. The minimum detectable energy is $E_{\min} = e\overline{N}V_{ph} = (1.6 \times 10^{-19}) \times 20 \times 2.0 = 6.4 \times 10^{-18}$ joule and the minimum detectable power is

$$P_{\min} = \frac{E_{\min}}{\tau} = \frac{6.4 \times 10^{-18}}{5} = 1.28 \times 10^{-18}\ \text{W}$$

4

TYPICAL DISTRIBUTION FUNCTIONS: VARIANCE THEOREM

In this chapter we discuss three typical distribution functions occurring in physics and engineering: (a) the binomial law, (b) Poisson's law, and (c) the normal law. In addition, we discuss Burgess's variance theorem for calculating averages in complicated statistical problems.

4.1 TYPICAL DISTRIBUTION FUNCTIONS

4.1a Binomial Process

A binomial process is defined as follows. Let an experiment be tried m times. Let p be the probability that the experiment is sucessful and $1-p$ the probability that it fails. Let there be n successes for the m trials, then

$$\overline{n} = mp; \qquad \operatorname{var} n = mp(1-p) \tag{4.1}$$

and the probability distribution is

$$P_m(n) = \frac{m!}{n!(m-n)!} p^n (1-p)^{m-n} \tag{4.2}$$

This is called the binomial law, since $P_m(n)$ is the $(n+1)$th term in the binomial expansion of $[p + (1-p)]^m$.

Before proving these theorems, we first discuss a few cases to which it applies.

1. ***Vacuum Pentode.*** "Trial" is the emission of an electron by the cathode, "success" means that it arrives at the anode, and "failure" means that it is intercepted by the screen grid.

2. ***Transistor.*** "Trial" is the injection of a minority carrier from the emitter into the base, "success" means that it is collected by the collector, and "failure" means that it recombines in the base.

3. ***Photodiode.*** "Trial" is the absorption of a photon by the $p-n$ junction, "success" is when the generated hole–electron pair is collected, and "failure" means that the pair is not collected.

4. ***Photoemissive Cathode.*** "Trial" is the absorption of a photon by the cathode, "success" means that a photoelectron is emitted, and "failure" means that no electron is emitted.

5. ***Partly Silvered Mirror.*** "Trial" is the arrival of a photon at the mirror, "success" means that it is transmitted, and "failure" means that it is reflected.

Next a few words about (4.1). The first part is obvious. Since p is the probability of success and the trials are independent, $\bar{n}$ is equal to the number of trials times the probability of success per trial, as stated. The second part is understandable by bearing in mind that success and failure are interchangeable concepts, and hence $\operatorname{var} n$ must be symmetrical in p and $(1-p)$; the second half of (4.1) is the simplest expression that does so. Because individual trials are independent, it is also obvious that $\operatorname{var} n$ must be proportional to m; hence it is sufficient to prove the second half of (4.1) for $m=1$. However, that is simple, for if $m=1$, then n is either 0 or 1. Now if $n=0$, then $n^2=0$ and if $n=1$, then $n^2=1$, so that $n^2=n$ in either case. Hence

$$\bar{n}=p; \qquad \overline{n^2}=\bar{n}=p$$

$$\operatorname{var} n=\overline{n^2}-(\bar{n})^2=p-p^2=p(1-p)$$

as had to be proved.

It should be noted that $\operatorname{var} n=0$ for $p=0$ and $p=1$ and that it has its maximum value for $p=\frac{1}{2}$, in which case $\operatorname{var} n=\frac{1}{4}m$.

Next we prove (4.2). The probability that m trials will just give n successes and $m-n$ failures in a prescribed order is $p^n(1-p)^{m-n}$. Yet there are

$$\frac{m!}{n!(m-n)!}$$

ways of assigning the n successes to possible positions in the series of m trials and all are equally probable. Hence (4.2) is correct.

Another important example of binomial statistics is Fermi statistics. In Fermi statistics an "energy state" is described by quantum numbers n_1, n_2, n_3 and m_s, where n_1, n_2, n_3 describe the wave pattern and m_s describes the spin orientation. According to Pauli's exclusion principle an energy state is either empty or occupied by one electron. Hence binomial statistics applies. If f is the probability that the energy state is occupied and n is the number of electrons in that state, then

$$\bar{n}=f; \qquad \operatorname{var} n=f(1-f)=\bar{n}-(\bar{n})^2 \tag{4.3}$$

There is nothing mysterious about the latter formula; it is a simple consequence of Pauli's exclusion principle.

4.1b Poisson Process

The Poisson process can be stated as follows. Let a number of events occur independently and at random instants. If n events occur in a particular time interval τ, then

$$\operatorname{var} n=\bar{n} \tag{4.4}$$

and the distribution function $P(n)$ for n is the *Poisson distribution*

$$P(n)=\frac{(\bar{n})^n}{n!}\exp(-\bar{n}) \tag{4.5}$$

Before proving these theorems, a few examples of such a process will be given.

1. ***Saturation Current of a Thermionic Cathode.*** The electrons are emitted independently and at random instants. Hence the electron emission is a Poisson process.

2. ***Injection of Electrons into the* P region of an n^+-p diode.** The electrons are injected independently and at random instants. Hence electron injection is a Poisson process.

3. ***Emission of Photons by a Laser.*** The photons are emitted independently and at random instants; hence this emission of photons is a Poisson process. Since the arrival of photons coming from a laser is also a series of independent events occurring at random instants, the arrival of photons coming from a laser is also a Poisson process.

However, the emission of black-body radiation is not a Poisson process, since the photons are emitted in bunches. Consequently, $\operatorname{var} n>\bar{n}$; the emission process is then said to be *superpoissonian.*

To prove (4.4) and (4.5) we divide the interval τ into m time intervals of equal length τ/m, where $m \gg \bar{n}$. Then the probability $p=(\bar{n}/m)$ that a single event occurs during a time interval τ/m is very small and the probability that more than one event occurs is negligible. Therefore, the binomial law holds, or

$$\bar{n} = mp; \qquad \operatorname{var} n = mp(1-p)$$

and the distribution function is $P_m(n)$. We now let m go to infinity such that $mp=\bar{n}$ remains constant. Then $p \to 0$ and hence $\operatorname{var} n = mp = \bar{n}$, as had to be proved.

To prove (4.5) we must prove that

$$\lim_{m\to\infty, mp=\bar{n}} P_m(n) = \frac{(\bar{n})^n}{n!}\exp(-\bar{n})$$

The proof is simple. We write

$$P_m(n) = \frac{(1-p)^m}{n!}\frac{m!}{(m-n)!}(1-p)^{-n}p^n$$

Now

$$(1-p)^m = \left[(1-p)^{1/p}\right]^{mp} = \left[(1-p)^{1/p}\right]^{\bar{n}} = \left(\frac{1}{e}\right)^{\bar{n}} = \exp(-\bar{n})$$

since $\lim_{p\to 0}(1-p)^{1/p} = 1/e$. Furthermore $\lim_{p\to 0}(1-p)^{-n} = 1$ for any finite n. Therefore we must prove

$$\lim_{m\to\infty, mp=\bar{n}} \frac{m!p^n}{(m-n)!} = (\bar{n})^n$$

Now since $p = (\bar{n}/m)$,

$$\lim_{m\to\infty, mp=\bar{n}} \frac{m!}{(m-n)!}p^n = \lim_{m\to\infty}\frac{m!}{(m-n)!}\frac{(\bar{n})^n}{m^n}$$

$$\lim_{m\to\infty}\frac{m}{m}\cdot\frac{m-1}{m}\cdot\frac{m-2}{m}\cdots\frac{m-n+1}{m}\cdot(\bar{n})^n = (\bar{n})^n$$

as had to be proved.

We now apply Poisson's law to the secondary emission dynode to evaluate the probability that no secondary electron is emitted. We assume that the secondary multiplication process may approximately be represented by a Poisson process as long as the primary energy is not too large

(see Section 4.2). Hence the probability that no electron is emitted is

$$P(0) = 1 - p = \frac{(\bar{n})^0}{0!}\exp(-\bar{n}) = \exp(-\bar{n}) = \exp(-\delta)$$

or $p = 1 - P(0) = 1 - \exp(-\delta)$. This proves (3.14).

4.1c The Normal Law

For large values of n the binomial distribution $P_m(n)$ and the Poisson distribution $P(n)$ can be approximated by the so-called *normal distribution*

$$P(n) = \frac{1}{(2\pi \operatorname{var} n)^{1/2}}\exp\left[-\frac{(n-\bar{n})^2}{2\operatorname{var} n}\right] \tag{4.6}$$

The case $\operatorname{var} n = \bar{n}$ is sometimes called the *Gaussian* distribution. For the Poisson (or shot noise) process, $\operatorname{var} n = \bar{n}$.

The proof of (4.6) proceeds as follows. Let $P_m(n)$ have a maximum value at $n = n_0 \cong \bar{n}$. We now make a Taylor expansion of $\ln P_m(n)$ with respect to $\Delta n = (n - n_0)$ and terminate with the second-order term; this is sufficiently accurate if n_0 is large. Since n is a large number, we may define

$$\frac{d}{dn}\left[\ln P_m(n)\right] \cong \frac{\ln P_m(n+1) - \ln P_m(n)}{n+1-n} \tag{4.6a}$$

Setting this equal to zero, we find n_0. Evaluating the second derivative at $n = n_0$ by differentiation, we have

$$\ln P_m(n) = \ln P_m(n_0) + \frac{1}{2}\frac{d^2}{dn^2}\ln P_m(n)\bigg|_{n=n_0}\Delta n^2 \tag{4.6b}$$

Comparing this to (4.6) shows complete identity and yields $\operatorname{var} n$.

We can have a similar distribution for a continuous random variable x with average value $\bar{x}$ and variance $\sigma_x^2 = \overline{x^2} - (\bar{x})^2$. In analogy with (4.6)

$$P(x) = \frac{1}{(2\pi\sigma_x^2)^{1/2}}\exp\left[-\frac{(x-\bar{x})^2}{2\sigma_x^2}\right] \tag{4.7}$$

This is easily extended to the case of more variables. In the case of two variables x and y one must consider the averages $\overline{x^2}$, $\overline{xy}$, and $\overline{y^2}$. We first discuss the case $\overline{xy} = 0$. The two variables are then independent and one has the joint-probability density function

$$P(x,y)=\frac{1}{2\pi\left(\sigma_x^2\sigma_y^2\right)^{1/2}}\exp\left[-\frac{\left(x-\bar{x}\right)^2}{2\sigma_x^2}-\frac{\left(y-\bar{y}\right)^2}{2\sigma_y^2}\right] \tag{4.8}$$

One can now make an orthogonal transformation to new random variables x_1 and x_2 that may be partially correlated. One then obtains*

$$P(x_1,x_2)=\frac{1}{2\pi M^{1/2}}\exp\left[-\tfrac{1}{2}(\mathbf{x}'\mathbf{M}^{-1}\mathbf{x})\right] \tag{4.9}$$

where x, x', and $\mathbf{M}$ are matrices

$$\mathbf{x}=\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix};\qquad \mathbf{x}'=\{x_1,x_2\};\qquad \mathbf{M}=\begin{Bmatrix} \overline{x_1^2} & \overline{x_1x_2} \\ \overline{x_1x_2} & \overline{x_2^2} \end{Bmatrix} \tag{4.9a}$$

$\mathbf{M}^{-1}$ is the reciprocal of the matrix $\mathbf{M}$ and M is the determinant of $\mathbf{M}$.

This is easily extended to more variables.*

It turns out that practically all noise phenomena encountered in physics and engineering can be described by the normal distribution. This is a consequence of the *central limit theorem*, which holds for practically all noise phenomena. It can be formulated as follows:†

If $X_1, X_2 \ldots X_n$ are independent random variables, all having the same probability density function, and hence equal averages $\overline{X_i}=\overline{X_1}$ and equal variances $\operatorname{var}X_i=\operatorname{var}X_1=\sigma_1^2$, then the sum $Y=\Sigma_{l=1}^{n}X_l$ is asymptotically normal for large n, with an average $\overline{Y}=n\overline{X_1}$ and variance $\sigma_1^2 n$, if σ_1 exists.

As a consequence very little information can be gained by measuring the distribution function for practically all noise phenomena, since it will most likely be the normal distribution anyway.

4.2 THE VARIANCE THEOREM

There is a large class of noise problems that can be described by the following process, identified by Burgess.*

Let a sequence of N events occur during a time interval τ. Let a quantity a_i be assigned to each event. Let a second quantity n be defined by

$$n=\sum_{i=1}^{N}a_i \tag{4.10}$$

*A. van der Ziel, *Noise*, Prentice Hall, Englewood Cliffs, N. J., 1954, Appendix 1.
†H. Cramér, *Mathematical Methods of Statistics*, Princeton U. P., 1946.
*R. E. Burgess, *Faraday Soc. Disc.* **28**, 151 (1959).

Let now N and a_i both fluctuate and let $\overline{a_i}=\overline{a}$ and $\overline{a_i^2}=\overline{a^2}$ be independent of i. Then

$$\overline{n}=\overline{N}\,\overline{a}\,;\qquad \operatorname{var}n=\left(\overline{a}\right)^2\operatorname{var}N+\overline{N}\operatorname{var}a \tag{4.11}$$

(Burgess's variance theorem).

To prove the variance theorem we start with an ensemble and average over it in three steps:

1. We divide the ensemble into subensembles with the same value of N.
2. We first average over each subensemble separately.
3. We finally average over all subensembles.

We shall denote a subensemble average by $\overline{}^{s}$ and the ensemble average by $\overline{}$. We then have $\overline{n}=\overline{\overline{n}^s}=\overline{N\overline{a}}=\overline{N}\overline{a}$, which proves the first half of (4.11). To prove the second half we observe that

$$n^2=\sum_{i=1}^{N}\sum_{j=1}^{N}a_ia_j$$

Hence

$$\overline{n^2}^{s}=\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{a_ia_j}^{s}=N(N-1)\left(\overline{a}\right)^2+N\overline{a^2}$$

since there are N terms a_ia_j with $i=j$ for which $\overline{a_ia_j}^{s}=\overline{a^2}$ and $N(N-1)$ terms a_ia_j with $i\neq j$ for which $\overline{a_ia_j}^{s}=(\overline{a})^2$. Hence

$$\overline{n^2}^{s}=N^2\left(\overline{a}\right)^2+N\operatorname{var}a \quad \text{or}$$

$$\overline{n^2}=\overline{\overline{n^2}^{s}}=\overline{N^2}\left(\overline{a}\right)^2+\overline{N}\operatorname{var}a$$

so that

$$\operatorname{var}n=\overline{n^2}-\left(\overline{n}\right)^2=\left(\overline{a}\right)^2\left[\overline{N^2}-\left(\overline{N}\right)^2\right]+\overline{N}\operatorname{var}a$$

which proves the second half of (4.11).

4.2a Applications to Pentodes and Transistors: Partition Noise

As a first example we consider the *vacuum pentode*. Let n_c electrons be emitted by the cathode; let n_a arrive at the anode and n_2 at the screen grid. Under space-charge limited conditions in the cathode-grid region, the cathode-current fluctuations are smoothed by the space charge in front of the cathode, resulting in $\operatorname{var} n_c < \overline{n_c}$.

For this case n_c corresponds to N. Let $a_i = 1$ if the electron arrives at the anode and $a_i = 0$ if the electron arrives at the screen grid. If λ is the probability that the electron arrives at the anode, then $\bar{a} = \lambda$ and $\operatorname{var} a = \lambda(1-\lambda)$. Hence the variance theorem applies and

$$\overline{n_a} = \overline{n_c}\,\lambda; \qquad \overline{n_2} = \overline{n_c}\,(1-\lambda) \tag{4.12}$$

$$\operatorname{var} n_a = \lambda^2 \operatorname{var} n_c + \overline{n_c}\,\lambda(1-\lambda); \qquad \operatorname{var} n_2 = (1-\lambda)^2 \operatorname{var} n_c + \overline{n_c}\,\lambda(1-\lambda) \tag{4.13}$$

The term $\lambda^2 \operatorname{var} n_c$ is called *attenuated shot noise* and the term $\overline{n_c}\lambda(1-\lambda)$ is called *partition noise*.

We can rewrite the first half of (4.13) as

$$\operatorname{var} n_a = \overline{n_c}\,\lambda + \lambda^2(\operatorname{var} n_c - \overline{n_c}) = \overline{n_a} + \lambda^2(\operatorname{var} n_c - \overline{n_c}) \tag{4.13a}$$

Hence if the cathode noise is not attenuated by space charge, or $\operatorname{var} n_c = \overline{n_c}$, then $\operatorname{var} n_a = \overline{n_a}$. In other words the anode current has full-shot noise if the cathode current has full-shot noise.

We can also write

$$n_c = n_a + n_2 \qquad \Delta n_c = \Delta n_a + \Delta n_2$$

$$\overline{\Delta n_c^2} = \operatorname{var} n_c = \overline{\Delta n_a^2} + 2\,\overline{\Delta n_a \Delta n_2} + \overline{\Delta n_2^2}$$

$$= \operatorname{var} n_a + \operatorname{var} n_2 + 2\,\overline{\Delta n_a \Delta n_2}$$

$$= \left[\lambda^2 + (1-\lambda)^2\right]\operatorname{var} n_c + 2\,\overline{n_c}\,\lambda(1-\lambda) + 2\,\overline{\Delta n_a \Delta n_2}$$

Hence since $1 - \lambda^2 - (1-\lambda)^2 = 2\lambda(1-\lambda)$

$$2\,\overline{\Delta n_a \Delta n_2} = -2\,\overline{n_c}\,\lambda(1-\lambda) + 2\lambda(1-\lambda)\operatorname{var} n_c$$

or

$$\overline{\Delta n_a \Delta n_2} = (\operatorname{var} n_c - \overline{n_c})\lambda(1-\lambda) \tag{4.14}$$

Therefore, if $\operatorname{var} n_c = \overline{n_c}$, Δn_a, and Δn_2 are independent.

We can also write this as follows: Put

$$\Delta n_a = \lambda \Delta n_c + \Delta n_p; \qquad \Delta n_2 = (1-\lambda)\Delta n_c - \Delta n_p$$

where Δn_p is independent of Δn_c. This set of equations makes sense, for an electron going to the screen grid is missed at the anode and vice versa. Hence according to (4.13)

$$\operatorname{var} n_a = \lambda^2\,\overline{\Delta n_c^2} + \overline{\Delta n_p^2}, \qquad \text{or} \qquad \overline{\Delta n_p^2} = \overline{n_c}\,\lambda(1-\lambda) \tag{4.15}$$

and

$$\overline{\Delta n_a \Delta n_2} = \lambda(1-\lambda)\,\overline{\Delta n_c^2} - \overline{\Delta n_p^2} = \lambda(1-\lambda)(\operatorname{var} n_c - \overline{n_c})$$

in agreement with (4.14). The term $\overline{\Delta n_p^2}$ is called *partition noise*; it is the only noise observed when the cathode current fluctuations are fully smoothed out.

As a second example we consider the *transistor*. Let n_E be the rate of electron injection into the base by the emitter; n_B the rate of recombination in the base; and n_C the rate of electron collection by the collector. Then the variance theorem holds and

$$n_E = n_B + n_C; \qquad \overline{n_B} = (1-\alpha_F)\,\overline{n_E}; \qquad \overline{n_C} = \alpha_F\,\overline{n_E} \tag{4.16}$$

where α_F is the d.c. current amplification factor of the transistor. Moreover, since $\operatorname{var} n_E = \overline{n_E}$,

$$\operatorname{var} n_C = \alpha_F^2 \operatorname{var} n_E + \overline{n_E}\,\alpha_F(1-\alpha_F) = \overline{n_E}\,\alpha_F = \overline{n_C} \tag{4.17}$$

$$\operatorname{var} n_B = (1-\alpha_F)^2 \operatorname{var} n_E + \overline{n_E}\,\alpha_F(1-\alpha_F) = \overline{n_E}\,(1-\alpha_F) = \overline{n_B} \tag{4.18}$$

Finally, if $\Delta n_B = n_B - \overline{n_B}$ and $\Delta n_C = n_C - \overline{n_C}$,

$$\overline{\Delta n_B \Delta n_C} = (\operatorname{var} n_E - \overline{n_E})\alpha_F(1-\alpha_F) = 0 \tag{4.19}$$

in analogy with (4.14). In other words, in a transistor the collector and the base currents fluctuate independently and each current shows full shot noise.

4.2b Electron-Multiplication Processes

We now apply our results to an electron-multiplication process. Let N be the number of electrons arriving at the device and let $\operatorname{var} N = \overline{N}$ (Poisson process). Let a_i be the number of secondary electrons produced by the ith primary; since all primaries have equal probability as far as multiplication is concerned, $\overline{a_i} = \bar{a}$ and $\overline{a_i^2} = \overline{a^2}$, independent of i. Hence the variance theorem is valid and

$$\bar{n} = \overline{N}\,\bar{a}; \qquad \operatorname{var} n = (\bar{a})^2 \operatorname{var} N + \overline{N} \operatorname{var} a \tag{4.20}$$

The first term in $\operatorname{var} n$ is called *amplified primary noise* and the term $\overline{N}\ \operatorname{var} a$ is called *multiplication noise*. If we now make use of the fact that $\operatorname{var} N = \overline{N}$ we obtain

$$\operatorname{var} n = \overline{N}\ \overline{a^2} = \overline{N}\ \overline{a}\ \frac{\overline{a^2}}{\overline{a}} = \overline{n}\ \frac{\overline{a^2}}{\overline{a}} \tag{4.20a}$$

In the particular case that $\operatorname{var} a = \overline{a}$, we have $\overline{a^2} = \overline{a} + (\overline{a})^2$. Then

$$\operatorname{var} n = \overline{n}\,(\overline{a} + 1) \tag{4.20b}$$

In the case that by some method $\operatorname{var} N$ has been completely smoothed out, so that $\operatorname{var} N = 0$, only the multiplication noise is left, or

$$\operatorname{var} n = \overline{N}\ \operatorname{var} a \tag{4.20c}$$

We now apply this to hole–electron pair production in a $p-n$ junction, or to hole–electron pair production in a photoconductive cell, by high-energy quanta, electrons, or heavier particles. In that case the preceding theory is valid, and if $\operatorname{var} N = \overline{N}$ the noise is completely described by (4.20a).

We now consider the case of hole–electron pair production in $p-n$ junctions, in which the pairs have a probability λ of being collected. If m pairs are collected for n pairs produced, then, according to the variance theorem

$$\overline{m} = \overline{n}\lambda = \overline{N}\ \overline{a}\lambda \tag{4.21}$$

$$\operatorname{var} m = \lambda^2 \operatorname{var} n + \overline{n}\lambda(1-\lambda) = \overline{n}\lambda^2 \frac{\overline{a^2}}{\overline{a}} + \overline{n}\lambda(1-\lambda)$$

$$= \overline{m}\left(\lambda \frac{\overline{a^2}}{\overline{a}} + 1 - \lambda\right) \tag{4.22}$$

In the particular case that the multiplication process is also a Poisson process, we have $\operatorname{var} a = \overline{a} = \overline{a^2} - (\overline{a})^2$, or $\overline{a^2} = (\overline{a})^2 + \overline{a}$. Consequently, from (4.22)

$$\operatorname{var} m = \overline{m}\,(\lambda\,\overline{a} + 1) \tag{4.23}$$

Something should be said about the escape probability of produced photoelectrons in photocathodes.* Formerly this escape probability was

*A. van der Ziel, *Solid State Physical Electronics*, 3rd ed., Prentice Hall, Englewood Cliffs, N. J., 1976.

not very high because the electron affinity χ of the semiconductor surface was positive (Fig. 4.1*a*). Since only those photoelectrons can escape for which $\frac{1}{2}mv_x^2 > \chi$, where v_x is the velocity component perpendicular to the surface, the escape probability of the photoelectrons was quite small. Photocathodes with a negative electron affinity χ exist; in that case potentially all electrons that are brought into the conduction band can escape (Fig. 4.1*b*). The electrons can now lose their energy by pair production if $\frac{1}{2}mv^2 > eE_g$ and to lattice vibrations if $\frac{1}{2}mv^2 < eE_g$; here v is the velocity of the photoelectrons.

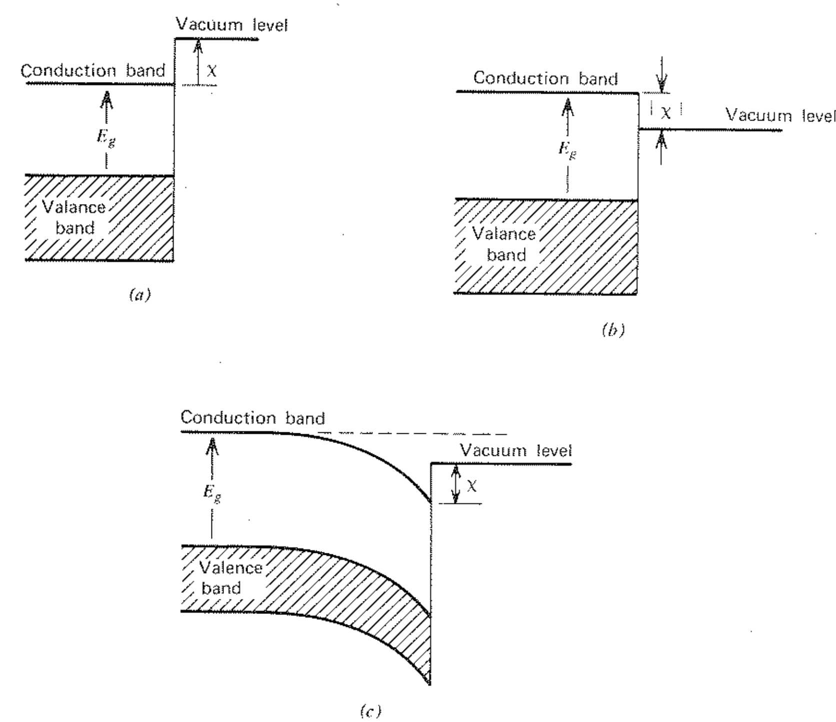


Figure 4.1. (*a*) Semiconductor band diagram with valence band, conduction band, gap width E_g, and positive electron affinity χ;(*b*) same but with negative electrons affinity χ;(*c*) same but with positive electron affinity χ and band bending near the surface, so that the bottom of the conduction band in the bulk is above the vacuum level.

To make the thermionic emission of the cathode as small as possible, one should make the work function of the semiconductor surface as large as possible. This is achieved by making the photocathode p^+ type. The photoelectrons along their way to the surface will now recombine with holes. If the electrons have a lifetime τ_n and a diffusion constant D_n, then their diffusion length $L_n=(D_n\tau_n)^{1/2}$. The travel to the surface is a diffusion process, and the escape depth of the photoelectrons equals the diffusion length L_n. This can be as large as several μm.

The case pictured in Fig. 4.1*b* is the so-called *flatband case*, where there is no band bending at the surface. This is the case for GaAs and GaP surfaces covered by an —O—Cs layer. In other cases, as in $In_xGa_{1-x}As$ or Si surfaces covered with an —O—Cs layer, there is band bending at the surface (Fig. 4.1*c*); the escape probability is now smaller than in the flatband case.

4.2c Secondary Electron Multiplication

We now apply the above considerations to secondary electron-emitting dynodes. In that case it is common practice to put $\bar{a}=\delta$ and $\overline{a^2}=\kappa\delta$. Equation (4.20) may now be written

$$\bar{n}=\bar{N}\delta; \qquad \operatorname{var} n=\delta^2 \operatorname{var} N+\bar{N}\delta(\kappa-\delta) \tag{4.24}$$

The second term is usually called *secondary emission noise*. If $\operatorname{var} N=\bar{N}$, then

$$\operatorname{var} n=\bar{n}\kappa \tag{4.24a}$$

Since $\kappa \gg 1$ for a good secondary emission-multiplier stage, $\operatorname{var} n \gg \bar{n}$; the noise is thus *super-Poissonian*.

Often, especially at relatively low primary energies, $(\kappa-\delta)\cong 1$ so that

$$\operatorname{var} n=\delta^2 \operatorname{var} N+\bar{n} \tag{4.24b}$$

This is *compatible* with the assumption that the secondary emission noise is a Poisson process, as was assumed in the derivation of (3.14) given in Section 4.1b, but does not prove it.

At higher primary energies δ passes through a maximum δ_{max} at the primary energy $E_{p0}=E_{max}$ and decreases for $E_{p0}>E_{max}$, whereas κ saturates beyond E_{max}.* If E_{p0} is so chosen that κ/δ is a minimum, the relative importance of the secondary emission noise is minimized.

The best secondary emitters are GaAs and GaP, covered with an O—Cs monolayer; they have rather high δ values at relatively low primary energies and a very high δ_{max} at a relatively high primary energy E_{max}. As mentioned before, these emitters should be of the heavily doped *p* type to increase the work function and hence lower the thermionic emission.

*C. B. Murray, *Physica*, **38**, 549 (1968).

4.2d The Fano Factor

In many high-energy particle or quantum detection systems one uses multiplication processes that produce a large number of electrons or hole–electron pairs for each individual incoming particle or quantum. One can then characterize the multiplication process by the average number $\bar{a}$ of particles produced by each elementary event and by the variance in that number, $\operatorname{var} a$.

If the multiplication process were a Poisson process, we would have $\operatorname{var} a=\bar{a}$; since the multiplication process must satisfy the constraint that the *fixed* energy E of the incoming particle or quantum is used to produce electrons or hole–electron pairs, $\operatorname{var} a$ may differ from $\bar{a}$. It is then common practice to introduce the Fano factor F of the multiplication process, defined by the relation

$$\operatorname{var} a=F\bar{a} \tag{4.25}$$

Usually $\bar{a}$ is proportional to the energy E of the incoming particles or quanta, and hence the height of the pulses due to individual events is a measure for the energy E. Since $\operatorname{var} a$ corresponds to a spread in the pulse height, it characterizes an uncertainty in the measured energy E. If one wants to resolve current pulses due to particles or quanta of slightly different energies, then $\operatorname{var} a$ and hence the Fano factor F should be as small as possible.

This book discusses a variety of multiplication processes in subsequent chapters. The Fano factors of these processes are dealt with in Chapter 17.

6

NOISE CHARACTERIZATION DEVICES AND AMPLIFIERS

This chapter discusses noise characterization of passive and active networks in terms of the equivalent noise resistance, equivalent noise conductance, equivalent noise temperature, and the noise figure of four-terminal networks, with applications to FETs and transistors (Section 6.1). Sections 6.2 and 6.3 discuss the noise figure of some simple FET and transistor circuits.

6.1 NOISE CHARACTERIZATION

6.1a Two-Terminal Networks

The noise in any passive or active two-terminal network at the temperature T can be represented by an e.m.f. e_n in series with the impedance $Z=(R+jX)$ of the network (Fig. 6.1a). We now equate

$$\overline{e_n^2}=4kTR_n\Delta f \tag{6.1}$$

where Δf is the frequency band for which the e.m.f. e_n is defined and call R_n the equivalent noise *resistance* of the network.

If the circuit is a linear passive circuit that has thermal noise sources at the temperature T only, then

$$R_n=R \tag{6.1a}$$

If other noise sources are present, R_n may differ from R. The same is true for devices that do not have thermal noise or that show thermal noise but are nonlinear.

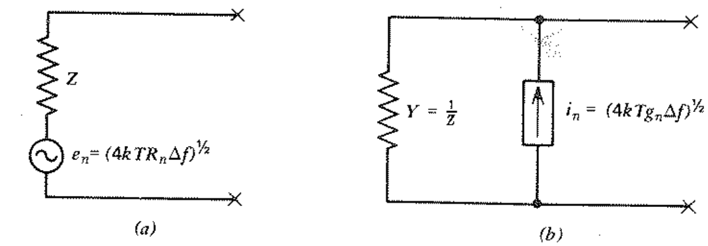


Figure 6.1. (a) Noise of a passive or active impedance Z, represented by an e.m.f. $(4kTR_n\Delta f)^{1/2}$ in series with Z; (b) noise of a passive or active admittance $Y=1/Z$, represented by a current generator $(4kTg_n\Delta f)^{1/2}$ in parallel with Y.

The noise in any passive or active two-terminal network at the temperature T can also be represented by a current generator i_n in parallel with the admittance $Y=1/Z=(g+jb)$ (Fig. 6.1b). We now equate

$$\overline{i_n^2}=4kTg_n\Delta f \tag{6.2}$$

and call g_n the *equivalent noise conductance* of the device.

If the circuit is a linear passive circuit, and all noise sources are thermal noise sources at the temperature T, then

$$g_n=g \tag{6.2a}$$

If other noise sources are present, or if the system is nonlinear, g_n may differ from g.

It is also possible to equate

$$\overline{i_n^2}=2eI_{eq}\Delta f \tag{6.3}$$

and call I_{eq} the *equivalent saturated diode* current of the network, This is especially appropriate if all noise sources are shot-noise sources. For example, in a saturated thermionic diode carrying a current I_d

$$I_{eq}=I_d \tag{6.3a}$$

In a space-charge-limited thermionic diode the current fluctuations are smoothed by the space charge. We expressed this by equating

$$\overline{i_n^2}=2e\Gamma^2I_d\Delta f$$

where Γ^2 is the space-charge suppression factor of the noise. Hence

$$I_{eq} = \Gamma^2 I_d, \quad \text{or} \quad \Gamma^2 = \frac{I_{eq}}{I_d} \tag{6.3b}$$

so that Γ^2 is easily determined.

Finally, it is possible to introduce an equivalent-noise temperature T_n in (6.1) and (6.2) by equating

$$\overline{e_n^2} = 4kT_n R\,\Delta f \quad \text{or} \quad T_n = \frac{R_n}{R}T \tag{6.4}$$

$$\overline{i_n^2} = 4kT_n g\,\Delta f \quad \text{or} \quad T_n = \frac{g_n}{g}T \tag{6.5}$$

This is useful in devices where the current carriers have an equivalent temperature different from the environment, and is exemplified in the case of hot-electron noise in solids or hot-electron noise in a gaseous plasma.

Since one may not know in advance which representation lends itself best to theoretical interpretation, one must be familiar with the conversion from one representation to another. For this and for the measurement of R_n, g_n, I_{eq} or T_n compare with van der Ziel.*

6.1b Four-Terminal Networks

Four-terminal devices like transistors and FETs generally have noise at both the output and the input. The noise must then be represented by an output-current generator i_0 and an input-current generator i_i that are partly correlated (Fig. 6.2*a*).

Let the input admittance of the device for short-circuited output be denoted by Y_i and the transfer admittance for short-circuited output by Y_m, then the equivalent circuit of the device can be represented by Fig. 6.2*b*, where the noise e.m.f. $e_n = (i_e / Y_m)$. It is now common practice to equate

$$\overline{e_n^2} = \frac{\overline{i_0^2}}{|Y_m|^2} = 4kTR_n\,\Delta f \tag{6.6}$$

and call R_n the *equivalent noise resistance* of the device.

* A. van der Ziel, *Noise: Sources, Characterization, Measurements,* Prentice Hall, Englewood Cliffs, N. J., 1970, Chapters 3–4.

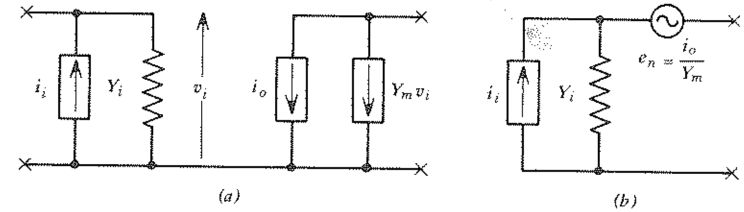


Figure 6.2. (*a*) Four-terminal active network in which the noise is represented by a current generator i_i in parallel to the input admittance Y_i and a current generator i_0 in parallel to the output, whereas the signal transfer properties are represented by a current generator $Y_m v_i$; (*b*) same network but the current generator i_0 is replaced by an e.m.f. $e_n = (i_0 / Y_m)$ in series with the input.

One can also equate

$$\overline{i_i^2} = 4kTg_{ni}\,\Delta f \tag{6.7}$$

and call g_{ni} the *equivalent input noise conductance* of the device.

Sometimes the correlation between i_0 and i_i must be taken into account.

We now apply this to FETs operating under saturated conditions. In the junction field effect transistor (JFET) the low-frequency noise is generation–recombination noise and the noise resistance is of the form

$$R_n = \frac{A}{1 + \omega^2\tau^2} \tag{6.8}$$

Typically, at room temperature A is of the order of $10^4\ \Omega$ for a good JFET and τ is of the order of 10^{-3} s. In the metal-oxide-semiconductor field effect transistor (MOSFET) the low-frequency noise is flicker noise and R_n is of the form

$$R_n = \frac{B}{f} \tag{6.8a}$$

Typically B is of the order of $10^8\ \Omega$ Hz but it can be much larger in poorer units.

At higher frequencies the noise is thermal noise of the conducting channel. For saturated devices a calculation shows that

$$\overline{i_0^2} = \overline{i_d^2} = \alpha \cdot 4kTg_m\,\Delta f \tag{6.9}$$

where g_m is the transconductance at saturation. Substituting (6.9) into (6.6) and putting $Y_m = g_m$ we have

$$R_n = \frac{\alpha}{g_m} \tag{6.9a}$$

The parameter α has a value 2/3 for MOSFETs and a value between 1/2 and 2/3 for JFETs, depending on bias. The value of α must be raised somewhat if there is a significant series resistance r_s on the source side of the channel. However, α never rises above unity, unless one has to do with hot-electron effects such as those which occur in very short channels.

At low frequencies there is a gate current I_g in JFETs of the order of 10^{-9}–10^{-12} Amp, and this current gives shot noise because the device is operated as a back-biased diode. It is now not very convenient to use a noise conductance g_{ni}; rather it is better to write

$$\overline{i_i^2} = \overline{i_g^2} = 2eI_g\Delta f \qquad (6.10)$$

The only exception is when the gate is floating. There are then two equal and opposite currents I_{g0} flowing across the gate junction and hence

$$\overline{i_i^2} = 4eI_{g0}\Delta f \qquad (6.10a)$$

Since the gate of the FET now acts as a floating diode, the input conductance is

$$g_i = \frac{1}{R_g} = \frac{eI_{g0}}{kT} \qquad (6.10b)$$

and hence

$$\overline{i_g^2} = 4kTg_i\Delta f \quad \text{or} \quad g_{ni} = g_i \qquad (6.10c)$$

For MOSFETs the leakage current is practically zero, so that the problem does not exist.

At high frequencies a gate noise exists both in the JFET and the MOSFET due to the capacitive coupling between channel and gate. A calculation shows that for saturation

$$\overline{i_i^2} = \overline{i_g^2} = \beta\cdot 4kTg_{gs}\Delta f \qquad (6.11)$$

where β lies between 1.0 and 4/3 for a JFET, depending on bias, and $\beta = 4/3$ for a MOSFET. Here g_{gs} is the input conductance of the device for short-circuited output; hence $g_{ni} = \beta g_{gs}$. A further calculation shows that for MOSFETs in saturation

$$g_{gs} = \frac{1}{5}\frac{\omega^2 C_{gs}^2}{g_m} \qquad (6.11a)$$

where C_{gs} is the gate-source capacitance of the device.

For some noise calculations one must take into account that i_g and i_d are somewhat correlated. A calculation shows that the correlation coefficient

$$c = \frac{\overline{i_g i_d^*}}{\left(\overline{i_g^2}\cdot\overline{i_d^2}\right)^{1/2}} \qquad (6.12)$$

is imaginary, except at the highest frequencies, and that $|c| \cong 0.40$ (more exactly, $|c| = 0.395$ for a MOSFET).

We have given here most of the formulas without proof. For details and further references see van der Ziel.* The equivalent circuit is shown in Fig. 6.3.

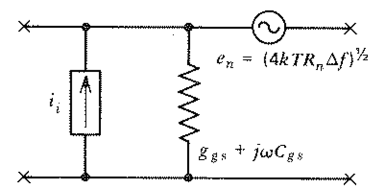


Figure 6.3. High-frequency equivalent noise circuit of a JFET or MOSFET in which the input noise is represented by a current generator i_i in parallel with the input admittance $Y_{gs} = g_{gs} + j\omega C_{gs}$ and the output noise is represented by an e.m.f. $e_n = (4kTR_n\Delta f)^{1/2}$ in series with the input.

We now turn to the transistor in common emitter connection (base as input and collector as output). At low frequencies there is some flicker noise with a $1/f$ spectrum; it can be represented by a current generator $i_i = i_{bf}$ between base and emitter where

$$\overline{i_i^2} = \overline{i_{bf}^2} = \frac{BI_B^\gamma}{f}\Delta f \qquad (6.13)$$

Here B is a constant, I_B is the base current, and γ is of the order of unity. This problem is discussed in Chapter 7.

At higher frequencies the noise is shot noise. If I_C is the collector current, then $g_m = (eI_C/kT)$ is the transconductance. Moreover we saw that

$$\overline{i_0^2} = \overline{i_c^2} = 2eI_C\Delta f = 2kTg_m\Delta f \qquad (6.14)$$

*A. van der Ziel, *Noise: Sources, Characterization, Measurements*, Prentice Hall, Englewood Cliffs, N. J. 1970.

was the collector noise. Hence

$$\overline{e_n^2} = \frac{\overline{i_c^2}}{g_m^2} = \frac{2kT\Delta f}{g_m} \quad (6.14a)$$

so that the noise resistance R_n of the device is

$$R_n = \frac{1}{2g_m} \quad (6.14b)$$

The base current I_B shows full-shot noise. The input conductance g_{be} for short-circuited output is equal to eI_B/kT and hence

$$\overline{i_i^2} = \overline{i_b^2} = 2eI_B\Delta f = 2kTg_{be}\Delta f \quad (6.15)$$

Consequently, the input-noise conductance is

$$g_{ni} = g_{be}/2 \quad (6.15a)$$

The equivalent circuit is shown in Fig. 6.4. To give an approximate high-frequency representation, the input capacitance C_{be} has been added. For a more detailed discussion see van der Ziel.*

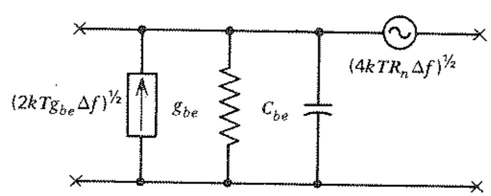


Figure 6.4. High-frequency equivalent noise circuit of a transistor in which the input noise is represented by a current generator $(2kTg_{be}\Delta f)^{1/2}$ in parallel with the input admittance $g_{be}+j\omega C_{be}$ and the output noise is represented by an e.m.f. $(4kTR_n\Delta f)^{1/2}$ in series with the input.

6.1c Noise Figure

The noise figure F of an amplifier is defined as

$$F = \frac{\text{Noise output power of amplifier}}{\text{Part due to the source resistance } R_s} \quad (6.16)$$

*A. van der Ziel, *Noise: Sources, Characterization, Measurements*, Prentice Hall, Englewood Cliffs, N. J., 1970.

We now refer all noise sources back to the input (Figs. 6.5*a*, *b*) and represent them by a current generator $(\overline{i_{eq}^2})^{1/2}$ in parallel with g_s or by a noise e.m.f. $(\overline{e_{eq}^2})^{1/2}$ in series with R_s. Then

$$F = \frac{\overline{i_{eq}^2}}{4kT\Delta f g_s} = \frac{\overline{e_{eq}^2}}{4kTR_s\Delta f} \quad (6.16a)$$

This suggests the following simple method for measuring noise figure. A saturated thermionic diode D is connected in parallel with R_s and so much current I_d is passed through D that the noise-output power of the amplifier is doubled. Then if $X_s = 0$,

$$\overline{i_{eq}^2} = 2eI_d\Delta f$$

and hence

$$F = \frac{e}{2kT} I_d R_s \quad (6.17)$$

We now calculate the noise figure of an FET amplifier stage at frequencies such that the gate noise is negligible. We thus have the circuit of Fig. 6.6*a*. Here R_s is the source impedance, which has thermal noise, and the device noise is characterized by the noise resistance R_n. Hence

$$F = \frac{4kTR_s\Delta f + 4kTR_n\Delta f}{4kTR_s\Delta f} = 1 + \frac{R_n}{R_s} \quad (6.18)$$

In practice one wants F to be as close to unity as possible. This means

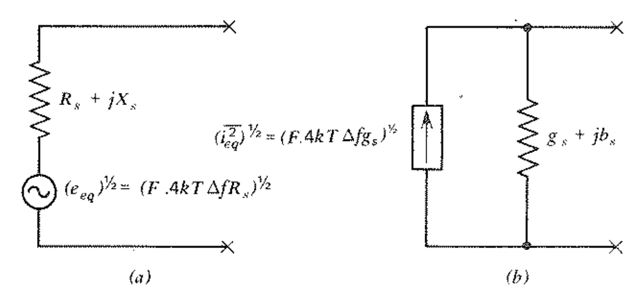


Figure 6.5. Definition of noise figure F: (*a*) amplifier noise represented by an e.m.f. $(\overline{e_{eq}^2})^{1/2} = (F\cdot 4kT\Delta fR_s)^{1/2}$ in series with the source impedance $Z_s = R_s + jX_s$; (b) amplifier noise represented by a current generator $(\overline{i_{eq}^2})^{1/2} = (F\cdot 4kT\Delta f g_s)^{1/2}$ in parallel to the source admittance $Y_s = (g_s + jb_s)$.

$R_s \gg R_n$. In some cases this cannot be done, because the input circuit may have to satisfy bandwidth restrictions.

In the JFET there is another reason why R_s cannot be made too large and that is that the effect of the gate leakage current I_g becomes important. We illustrate this in Fig. 6.6*b*. Here R_g is defined by the relation $1/R_g = (dI_g/dV_g)$. For large values fo R_s and R_g, of the order of 10^8–10^{12} Ω, the e.m.f. $(4kTR_n\Delta f)^{1/2}$ gives a negligible contribution to the noise figure and therefore it can be ignored. The noise figure F may therefore be written

$$F = \frac{4kT\Delta f/R_s + 2eI_g\Delta f}{4kT\Delta f/R_s} = 1 + \frac{e}{2kT}I_gR_s \cong 1 + 20I_gR_s \qquad (6.19)$$

For $I_g = 10^{-10}$ A and $R_s = 10^{10}$ Ω, $F = 21$. This means that R_s should be chosen smaller than 10^8 Ω in this case if one wants to make F close to unity.

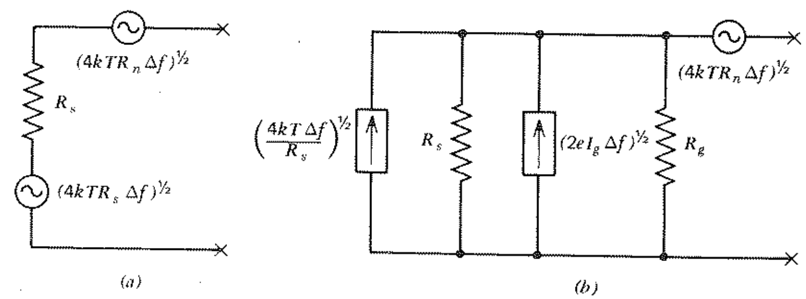


Figure 6.6. Noise-equivalent circuit of a JFET: (a) at low-impedance levels the device noise is represented by an e.m.f. $(4kTR_n\Delta f)^{1/2}$ in series with the input; (b) at high-impedance levels the device noise is represented by a current generator $(2eI_g\Delta f)^{1/2}$ in parallel to the differential input resistance R_g and the e.m.f. $(4kTR_n\Delta f)^{1/2}$ in series with the input.

In cascaded amplifiers the noise of subsequent stages must also be taken into account. Often the most important noise contribution is the contribution of the noise of the load resistance R_L in the output of the first stage. We now wish to point out that its effect is easily included with the help of the circuit of Fig. 6.7. Here the noise at the output of the device is represented by a current generator $(4kTR_n\Delta f g_m^2)^{1/2}$ and the noise of R_L by a current generator $(4kT\Delta f/R_L)^{1/2}$. We now define a new noise resistance R_n' by

$$4kTR_n'\Delta f g_m^2 = 4kTR_n\Delta f g_m^2 + \frac{4kT\Delta f}{R_L} \qquad (6.20)$$

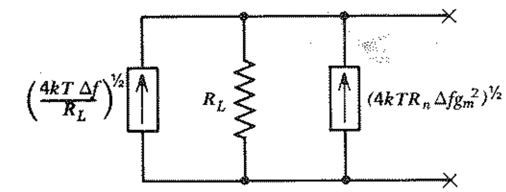


Figure 6.7. Evaluation of the effect of the thermal noise of the load resistance R_L in the output of an FET amplifier stage with an FET noise resistance R_n.

so that

$$R_n' = R_n + \frac{1}{g_m^2 R_L} \qquad (6.20a)$$

This expression must now be used in the noise-figure calculations; that is, (6.18) must be replaced by

$$F = 1 + \frac{R_n'}{R_s} = 1 + \frac{R_n}{R_s} + \frac{1}{g_m^2 R_s R_L} \qquad (6.21)$$

For example if $R_n = 100$ Ω, $R_s = 1000$ Ω, $R_L = 2000$ Ω, $g_m = 5.0$ m mhos, then $F = 1.12$ of which the part 0.02 comes from the last term in (6.21).

6.1d Friiss's Formula

When one wants to find the noise figure of several stages connected one behind the other (cascade connection), one must establish rules where the load resistance R_L of the interstage network must be counted. The general convention is to count the noise of R_L as belonging to the next stage. We followed this rule in the derivation of (6.18).

We must define the available power gain G_{av} of an amplifier stage. We define the available power P_{av_s} of a signal source consisting of an e.m.f. v_s in series with an internal resistance R_s as the power that can be fed into a matched load. This yields

$$P_{av_s} = \frac{1}{8}\frac{|v_s|^2}{R_s} \qquad (6.22)$$

The power gain G of an amplifier stage is now defined as

$$G = \frac{\text{Output power fed into load}}{\text{Power available at source}} \qquad (6.23)$$

If the load is matched to the output of the amplifier, the power gain is called the *available power gain*

$$G_{av} = \frac{\text{Output power fed into matched load}}{\text{Power available at source}} \tag{6.23a}$$

There is now an expression for the total noise figure F of the cascaded amplifier that holds if the following conditions are met:

1. The load resistance R_L of each interstage network is considered as belonging to the next stage.
2. The output conductance g_0 seen by each interstage network (i.e., seen when looking toward the preceding stage) is positive.
3. The ith stage, for the given coupling to the preceding stage, has a noise figure F_i and an available gain G_{av_i}.

If these three conditions are valid, the noise figure F of the combination is

$$F = 1 + (F_1 - 1) + \frac{F_2 - 1}{G_{av_1}} + \frac{F_3 - 1}{G_{av_1} G_{av_2}} \tag{6.24}$$

This is known as *Friiss's formula*. We do not use it for the following reasons:

1. It is very clumsy to handle when g_0 is near zero.
2. The circuits we discuss are sufficiently simple for direct evaluation.

6.2 APPLICATIONS TO FET CIRCUITS

6.2a Determination of R_n and I_g in FETs

We now discuss methods for determining the noise resistance R_n and for determining the gate current in JFETs. The circuit is shown in Fig. 6.8. Here the gate is connected to ground via a large bias resistance R_g. The gate is connected to a switch S by means of a blocking capacitor C_b of sufficiently low impedance. S has three positions:

POSITION 1. Connected to ground.

POSITION 2. Connected to a suitable chosen resistance R.

POSITION 3. Connected to a suitably chosen capacitor C.

The output of the FET feeds into a high-gain amplifier that is connected to a power meter.

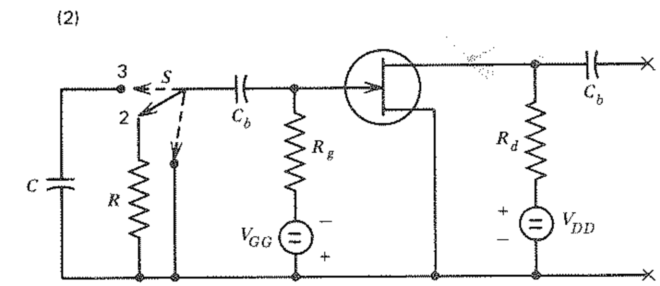


Figure 6.8. Measurement of noise resistance R_n and gate current I_g of a JFET: (*a*) by comparing the output noise for the switch S in positions 1 and 2 one can evaluate R_n; (b) by comparing the output noise for the switch S in positions 1 and 3 and comparing with the results of the previous measurement one can determine I_g.

We first discuss the measurement of R_n. Here R_g is so chosen that $R_g > 100R$. In position 1 the gate is short-circuited and the power meter reading M_1 corresponds to the noise of R_n, which has a mean square value $4kTR_n\Delta f$. In position 2 the switch is connected to the resistance R, and hence the equivalent noise at the input now has a mean square value $4kT(R+R_n)\Delta f$. If the power meter reading is now M_2, we have

$$\frac{M_2}{M_1} = \frac{4kT(R+R_n)\Delta f}{4kTR_n\Delta f} = 1 + \frac{R}{R_n}$$

so that

$$R_n = \frac{M_1}{M_2 - M_1} R \tag{6.25}$$

For a good measurement, M_1 and $(M_2 - M_1)$ should be comparable.

When one carries out the measurement, one obtains good agreement with theory at higher frequencies, but at lower frequencies the noise increases with decreasing frequency due to flicker noise in MOSFETs or generation–recombination noise in JFETs.

To measure the gate current I_g we make the resistance R_g much larger, for example, $R_g = 10^{10} - 10^{12}\ \Omega$. The gate current I_g gives a gate noise-current generator $(2eI_g\Delta f)^{1/2}$ in parallel to the input of the FET. If we now turn the switch in position 3, then the noise developed across C has a mean square value $\overline{v^2}$

$$\overline{v^2} = \frac{2eI_g\Delta f}{\omega^2 C^2} = 4kTR_n'\Delta f; \qquad R_n' = \frac{e}{2kT}\frac{I_g}{\omega^2 C^2} \tag{6.26}$$

If the output meter reading is now M_3, we have

$$\frac{R_n' + R_n}{R_n} = \frac{M_3}{M_1}; \qquad \frac{R_n'}{R_n} = \frac{M_3 - M_1}{M_1} \tag{6.26a}$$

Substituting for R_n we have

$$R_n' = \frac{M_3 - M_1}{M_2 - M_1} R \tag{6.26b}$$

It is now most suitable if M_3 and M_2 are comparable, and M_1 is small in comparison with M_3 and M_2. In that case

$$I_g = \frac{2kT}{e}\omega^2 C^2 R_n' \cong \frac{2kT}{e}\omega^2 C^2 \frac{M_3}{M_2} R \tag{6.26c}$$

Several conditions have to be met in order to keep the method simple:

1. The noise of $1/R$ must be large in comparison with the noise of I_g, or

$$\frac{4kT}{R} \gg 2eI_g \tag{6.27}$$

2. The noise of $1/R_g$ must be small in comparison with the noise of I_g, or

$$\frac{4kT}{R_g} \ll 2eI_g \tag{6.27a}$$

3. C must be so chosen that R_n' and R are comparable.

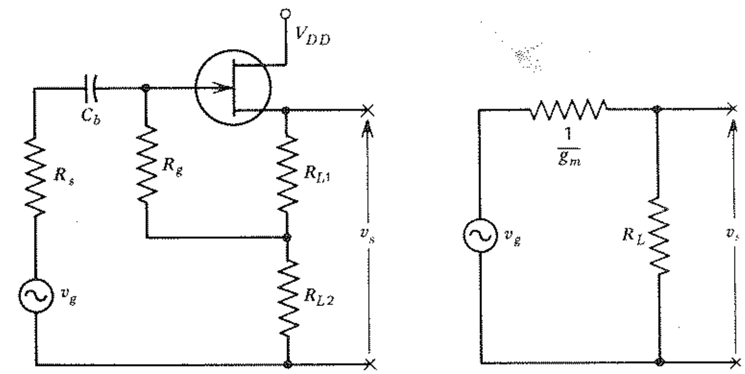
Example. $I_g = 10^{-11}$ A. Then the first term in (6.27) is 100 times as large as the second term if $R \cong 5 \times 10^7\ \Omega$, and the first term in (6.27a) is 100 times as small as the second term if $R_g \cong 5 \times 10^{11}\ \Omega$. If the frequency of measurement is 10 Hz, then $C = 30$ pF. If R is chosen smaller, C can be made larger.

6.2b The Source Follower and the Common-Gate FET

We first discuss the source follower. The circuit is shown in Fig. 6.9*a*. The load resistance R_L is split into two parts, R_{L1} and R_{L2}, to provide proper gate bias; the gate resistor R_g supplies that voltage to the gate. Figure 6.9*b* shows the equivalent circuit. We see from it that the voltage gain g_v is less than unity

$$g_v = \frac{v_s}{v_g} = \frac{R_L}{1/g_m + R_L} = \frac{g_m R_L}{1 + g_m R_L} \tag{6.28}$$

Here $R_L = (R_{L1} + R_{L2})$.



(*a*) (*b*)

Figure 6.9. (*a*) Circuit arrangement for a source follower; (*b*) equivalent circuit for the signal.

If the noise of R_g is neglected and R_n is the noise resistance of the FET, the equivalent circuit for the output circuit itself is shown in Fig. 6.7. Consequently, the noise resistance R_n' of the complete circuit is as given by (6.20a)

$$R_n' = R_n + \frac{1}{g_m^2 R_L} \tag{6.29}$$

and the noise figure of the complete circuit is

$$F = 1 + \frac{R_n'}{R_s} = 1 + \frac{R_n}{R_s} + \frac{1}{g_m^2 R_s R_L} \tag{6.30}$$

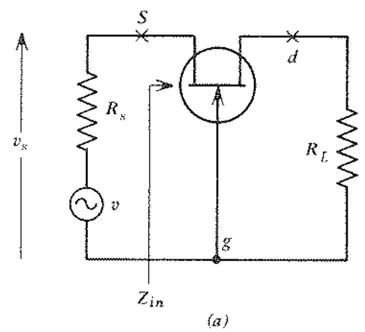
just as in (6.21).

Although the source follower has a voltage gain slightly less than unity, it serves the useful function of transforming from a high-impedance to low-impedance level with little loss of signal.

Next we turn to the common-gate circuit (Fig. 6.10*a*). It has the source as input electrode and the drain as output electrode. The signal source looks into an input impedance $Z_{in} = 1/g_m$, and the signal-transfer properties can be characterized by a current generator $g_m v_s$, where v_s is the a.c. voltage at the source electrode.

To simplify the discussion, we short-circuit the output; this is allowed since the noise of the load resistance R_L is counted as belonging to the next stage. First we consider the device noise i alone. It gives an input voltage

$$v_s = -i\frac{R_s \cdot 1/g_m}{R_s + 1/g_m} = -\frac{iR_s}{1 + g_m R_s} \tag{6.31}$$



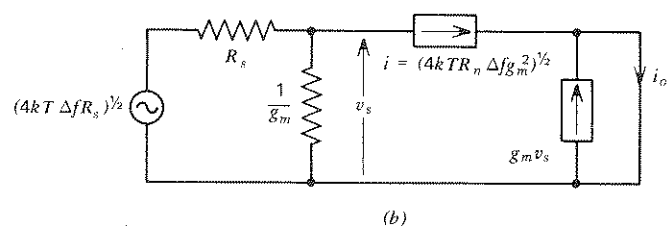


Figure 6.10. (*a*) Circuit arrangement for the common-gate circuit; (*b*) equivalent circuit of the common-gate circuit.

and hence a short-circuited output current

$$i_0' = i + g_m v_s = i\left(1 - \frac{g_m R_s}{1 + g_m R_s}\right) = \frac{i}{1 + g_m R_s} \qquad (6.31a)$$

Next we consider the noise of R_s. It gives an input voltage

$$v_s' = (4kTR_s\Delta f)^{1/2}\frac{1/g_m}{1/g_m + R_s} = \frac{(4kTR_s\Delta f)^{1/2}}{1 + g_m R_s} \qquad (6.32)$$

and hence a short-circuited output current $i''_0 = (4kTR_s\Delta f)^{1/2} g_m/(1 + g_m R_s)$. Hence adding both noise currents quadratically,

$$\overline{i_0^2} = (4kTR_s\Delta f + 4kTR_n\Delta f)\frac{g_m^2}{(1 + g_m R_s)^2} \qquad (6.33)$$

so that the noise figure is

$$F = 1 + \frac{R_n}{R_s} \qquad (6.33a)$$

just as for the common source circuit.

We notice that the effect of the current generator i is reduced by a factor $1 + g_m R_s$. This is important for understanding the dual gate FET circuit of Section 6.2d.

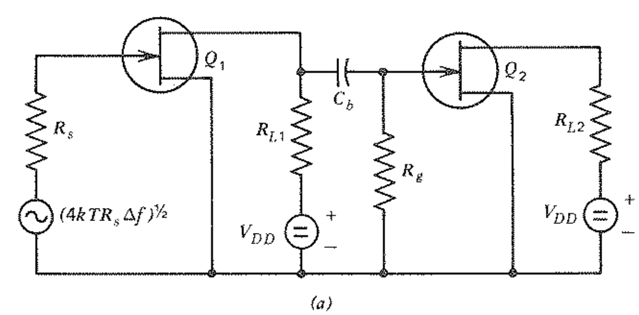
6.2c Noise of a Two-Stage FET Amplifier

The circuit is shown in Fig. 6.11*a*. We assume that $R_{L1} = R_{L2} = R_L$; to simplify matters we consider the output of the second stage short-circuited. We now calculate the noise resistance of $Q_1 + Q_2$. The interstage network plus the FETs can be represented by the equivalent circuit of Fig. 6.11*b*. We use this circuit to evaluate the noise resistance R_n' of the combination. We see by inspection that

$$\overline{v^2} = 4kTR_{n1}\Delta f g_m^2 R_L^2 + 4kTR_L\Delta f + 4kTR_{n2}\Delta f = 4kTR_n'\Delta f g_m^2 R_L^2 \qquad (6.34)$$

according to the definition of R_n'. Hence

$$R_n' = R_{n1} + \frac{R_L + R_{n2}}{g_m^2 R_L^2} \qquad (6.34a)$$



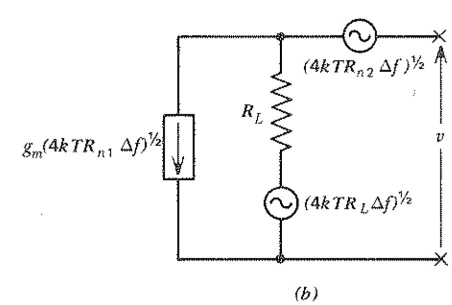


Figure 6.11. (*a*) Circuit arrangement for a two-stage FET circuit; (*b*) noise-equivalent circuit with all noise sources referred to the interstage network.

If $R_{n2} \ll R_L$ this reduces to

$$R_n' \cong R_{n1} + \frac{1}{g_m^2 R_L} \tag{6.34b}$$

so that only the effect of the load resistance R_L is important. The noise figure F is therefore

$$F = 1 + \frac{R_n'}{R_s} = 1 + \frac{1}{R_s}\left(R_{n1} + \frac{1}{g_m^2 R_L} + \frac{R_{n2}}{g_m^2 R_L^2}\right) \tag{6.35}$$

This equation would not have been so easily obtained from Friiss's formula.

6.2d The Dual-Gate FET

The dual-gate FET consists of two FETs Q_1 and Q_2 in a single envelope with the drain d_1 internally connected to the source s_2 so that only the electrodes g_1, s_1, g_2, and d_2 have external leads. For that reason the dual-gate FET is also called the *FET tetrode*. The circuit is shown in Fig. 6.12*a*. We assume that Q_1 and Q_2 are operated in saturation and that they are identical.

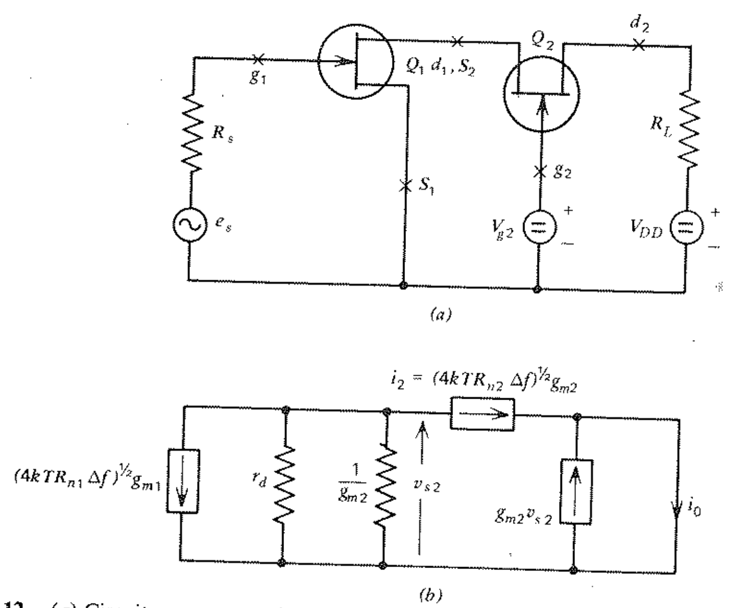


Figure 6.12. (*a*) Circuit arrangement for the dual-gate FET; (*b*) noise-equivalent circuit with the noise of the first half of the circuit referred to the output of that FET.

We now short-circuit the output and consider the noise of $Q_1 + Q_2$. We introduce the differential resistance r_d of Q_1; it is usually very large but we need it in the calculation. We now have the equivalent circuit of Fig. 6.12*b*, and use it to calculate the noise resistance of the combination. It is easily seen that

$$\overline{i_0^2} = 4kTR_{n1}\,\Delta f g_{m1}^2 \left(\frac{r_d}{1 + g_{m2} r_d}\right)^2 g_{m2}^2 + \frac{4kTR_{n2}\,\Delta f g_{m2}^2}{(1 + g_{m2} r_d)^2}$$

$$= 4kTR_n'\,\Delta f g_{m1}^2 \left(\frac{r_d}{1 + g_{m2} r_d}\right)^2 g_{m2}^2 \tag{6.36}$$

where $R_{n1} = R_{n2} = R_n$ are the noise resistances and $g_{m1} = g_{m2} = g_m$ the transconductances. Hence

$$R_n' = R_n\left[1 + \frac{1}{g_m^2 r_d^2}\right] \cong R_n \tag{6.36a}$$

if $g_m^2 r_d^2 \gg 1$, as is usually the case. Therefore, the dual-gate FET has the same noise resistance as the single FET, but it has a much smaller feedback capacitance between the output drain and the input gate than the single FET. Consequently it gives much better stability in high-frequency amplifiers.

6.2e The Pyroelectric Detector with a JFET Amplifier

A pyroelectric detector is a polarized ferroelectric capacitor C with a loss factor $\tan\delta$. If modulated radiation of frequency ω is incident upon the detector, an a.c. voltage of modulation frequency ω is developed across the capacitor (pyroelectric effect). Since the detector is a very high-impedance device, the voltage must be amplified by a high-impedance JFET amplifier. To ensure better stability it is recommended to operate the JFET with a floating base ($I_g = 0$, because two equal but opposite currents I_{g0} flow to the gate).

The pyroelectric detector can thus be used as a radiation detector. Its noise corresponds to thermal noise of its dielectric loss conductance

$$g = \omega C \tan\delta \tag{6.37}$$

The equivalent circuit of detector plus amplifier is shown in Fig. 6.13; here C_i is the input capacitance of the JFET for short-circuited output and $g_{g0} = (eI_{g0}/kT)$ is the input conductance of the JFET with floating gate. We assume that $\omega^2(C + C_i)^2 \gg (g + g_{g0})^2$; this is usually the case, since $(\tan\delta)^2 \ll 1$ and $g_{g0} \ll g$ except perhaps at the lowest frequencies. We then

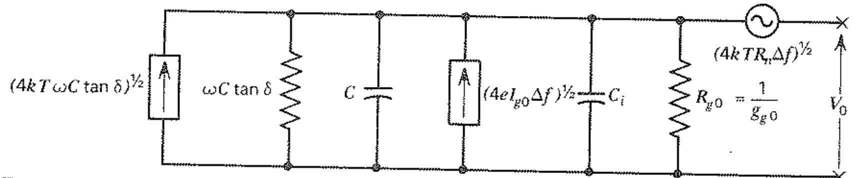


Figure 6.13. Pyroelectric detector having an equivalent circuit consisting of a capacitance C and a conductance $g=\omega C\tan\delta$ in parallel connected to the input of a JFET with floating gate.

have

$$\overline{v_0^2}=\frac{4kT\omega C\tan\delta\,\Delta f+4eI_{g0}\Delta f}{\omega^2(C+C_i)^2}+4kTR_n\Delta f=\frac{4kT\omega C\tan\delta\,\Delta f}{\omega^2(C+C_i)^2}\cdot F \quad (6.38)$$

where F is the noise figure of the detector. This definition makes sense, since the term

$$\frac{4kT\omega C\tan\delta\,\Delta f}{\omega^2(C+C_i)^2}$$

comes from the signal source. Hence

$$F=1+\frac{e}{kT}\frac{I_{g0}}{\omega C\tan\delta}+R_n\frac{\omega C}{\tan\delta}\left(\frac{C+C_i}{C}\right)^2 \quad (6.38a)$$

If we take $R_n=10^5\ \Omega$, $\omega=60/\text{sec}$, $C=20$ pF, $\tan\delta=0.01$, $C_i=5$ pF, $I_{g0}=10^{-12}$ A, then $F=4.35$; this value comes mostly from the second term and can be considerably reduced by applying Peltier cooling to the device.

For a MOSFET $I_{g0}\cong 0$, but R_n is usually much larger because of flicker effect. For example, if

$$R_n=10^7\ \Omega,\quad \omega=60/\text{sec},\quad C=20\text{ pF},\quad C_i=5\text{ pF},\quad \tan\delta=0.01,$$

then $F=2.g$.

In the case of a JFET the capacitance C can be so chosen that F is a minimum. This can be achieved by choosing a pyroelectric detector with the proper dielectric constant or electrode area.

6.2f Field-Effect Transistor Noise at High Frequencies

At high frequencies one must take into account both the gate noise and the drain noise. As a consequence there is now an optimum input tuning and

an optimum source conductance for minimum noise figure F_{min}. According to van der Ziel*

$$F_{min}=1+2R_ng_g+2\left(1.12R_ng_g+R_n^2g_g^2\right)^{1/2} \quad (6.39)$$

Since g_g varies as ω^2 over a wide frequency range and R_n is practically frequency independent in that range, $F_{min}-1$ varies as ω at lower frequencies and as ω^2 at higher frequencies. Furthermore his $g_n=\frac{4}{3}g_g(1-|c|^2)=1.12g_g$, since $|c|=0.395$.

6.3 NOISE IN TRANSISTOR CIRCUITS

6.3a Single Low-Frequency Stage

To find the noise figure at a single low-frequency stage, we modify the equivalent circuit of Fig. 6.4, and assume that the flicker noise is negligible. The modification consists in omitting the capacitance C_{be}, adding the base resistance r_b and its thermal noise and incorporating a source resistance R_s with its thermal noise into the circuit. We then obtain the equivalent circuit of Fig. 6.14*a*. It, in turn, can be replaced by the equivalent circuit of Fig. 6.14*b*.

As seen by inspection

$$\overline{v_0^2}=4kTR_s\Delta f\left(\frac{r_{be}}{R_s+r_b+r_{be}}\right)^2+4kTr_b\Delta f\left(\frac{r_{be}}{R_s+r_b+r_{be}}\right)^2+2kTr_{be}\Delta f\frac{(R_s+r_b)^2}{(R_s+r_b+r_{be})^2}+\frac{2kT\Delta f}{g_m} \quad (6.40)$$

so that the noise figure of the circuit is

$$F=1+\frac{r_b}{R_s}+\frac{1}{2}\frac{(R_s+r_b)^2}{R_sr_{be}}+\frac{1}{2g_mR_s}\left(\frac{R_s+r_b+r_{be}}{r_{be}}\right)^2 \quad (6.40a)$$

Usually $r_b\ll R_s$ and $R_s+r_b\ll r_{be}$, so that approximately

$$F\cong 1+\frac{r_b}{R_s}+\frac{1}{2}\frac{R_s}{r_{be}}+\frac{1}{2g_mR_s}=1+\frac{(1+2g_mr_b)}{2g_mR_s}+\frac{g_m}{2\beta_F}R_s \quad (6.41)$$

*A. van der Ziel, *Noise: Sources, Characterization, Measurements*, Prentice Hall, Englewood Cliffs, N. J., 1970. This neglects the effects of the correlation conductance g_{cor} and the tuned circuit conductance g_c.

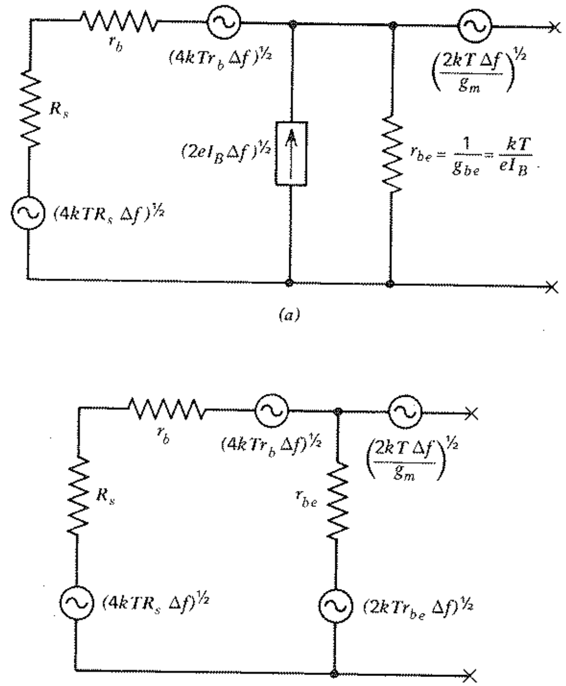


Figure 6.14. (*a*) Equivalent noise circuit of a transistor in common-emitter connection; (*b*) alternate equivalent circuit in which the current generator $(2eI_b\Delta f)^{1/2}$ in parallel with r_{be} is replaced by a noise e.m.f. $(2kTr_{be}\Delta f)^{1/2}$ in series with r_{be}.

where $g_m r_{be} = \beta_F$ is the current-amplification factor, so that $r_{be} = (\beta_F/g_m)$. This has a minimum value

$$F_{\min} = 1 + 2\left[\frac{(1+2g_m r_b)}{2g_m}\cdot\frac{g_m}{2\beta_F}\right]^{1/2} = 1 + \left(\frac{1+2g_m r_b}{\beta_F}\right)^{1/2} \qquad (6.41a)$$

for

$$R_s = (R_s)_{\text{opt}} = \left[\frac{(1+2g_m r_{be})}{2g_m}\cdot\frac{2\beta_F}{g_m}\right]^{1/2} = \frac{1}{g_m}\left[\beta_F(1+2g_m r_b)\right]^{1/2} \qquad (6.41b)$$

An accurate calculation, based on (6.40a), yields

$$F_{\min} = 1 + \frac{1+x}{\beta_F} + \left[\frac{1+2x}{\beta_F} + \left(\frac{1+x}{\beta_F}\right)^2\right]^{1/2} \qquad (6.42)$$

where

$$x = g_m r_b\left(1 + \frac{1}{\beta_F}\right) \qquad (6.42a)$$

For large β_F this reduces to (6.41a).

Example. If $g_m = 40$ m mhos, $r_b = 50\ \Omega$, $\beta_F = 100$, find $F_{\min}$ and $(R_s)_{\text{opt}}$ from (6.41a) and (6.41b) and evaluate the difference between (6.42) and (6.41a).

Answer. $F_{\min} = 1 + (0.05)^{1/2} = 1.22$, since $g_m r_b = 2.0$

$$(R_s)_{\text{opt}} = \frac{1}{40\times 10^{-3}}(500)^{1/2} = 560\ \Omega$$

From (6.42), $F_{\min} = 1.03 + (0.0509)^{1/2} = 1.26$, which is reasonably close.

We now draw the following conclusions:

1. The noise resistance of the circuit for zero source impedance is

$$R_{n0} = \lim_{R_s\to 0} R_s F = r_b + \frac{1}{2g_m} \qquad (6.43)$$

according to (6.41). That is, if one wants to measure the noise developed across a low impedance, one should use transistors with a very low base resistance r_b and a relatively large transconductance g_m (i.e., high currents). Noise resistances R_{n0} as low as 50 Ω can be obtained in this manner.

2. The noise can be represented by an input current generator $(\overline{i_{\text{eq}}^2})^{1/2}$ at the input, where

$$\overline{i_{\text{eq}}^2} = \frac{4kT\Delta f}{R_s}F \qquad (6.44)$$

We need this expression for deriving the noise figure of a two-stage transistor amplifier.

3. We can use the argument based on Fig. 6.7 to evaluate the noise resistance R_n' when the effect of the load resistance R_L is taken into account. With $R_L = 0$ the noise resistance is $1/2g_m$, and hence according to (6.29)

$$R_n' = R_n + \frac{1}{g_m^2 R_L} = \frac{1}{2g_m}\left(1 + \frac{2}{g_m R_L}\right) \qquad (6.45)$$

Therefore, in (6.40) we must replace g_m by $g_m/[1+2/(g_m R_L)]$, and hence in (6.41)

$$F \cong 1 + \frac{1+2/(g_m R_L) + 2g_m r_b}{2g_m R_s} + \frac{g_m R_s}{2\beta_F} \qquad (6.45a)$$

so that

$$F_{\min} = 1 + \left[\frac{1 + 2/(g_m R_L) + 2g_m r_b}{\beta_F}\right]^{1/2} \tag{6.45b}$$

Often the effect of the load resistance R_L is quite small and no great mistake is made if the term $2/(g_m R_L)$ is omitted.

6.3b Two-Stage Low-Frequency Transistor Amplifier

The circuit is shown in Fig. 6.15; for the sake of simplicity the output of the second stage is short-circuited and the bias circuits are omitted.

According to the previous section the noise of the second stage is given by an equivalent current generator $[F_2(4kT\Delta f)/R_L]^{1/2}$ where

$$F_2 \cong 1 + \frac{R_L}{2r_{be}} + \frac{1 + 2g_m r_b}{2g_m R_L} \tag{6.46}$$

Therefore, in (6.40) we must multiply the last term by the factor

$$1 + 2F_2/(g_m R_L) \tag{6.46a}$$

and hence in (6.41)

$$F = 1 + \frac{1 + 2F_2/(g_m R_L) + 2g_m r_b}{2g_m R_s} + \frac{g_m R_s}{2\beta_F} \tag{6.47}$$

so that

$$F_{\min} = 1 + \left[\frac{1 + 2F_2/(g_m R_L) + 2g_m r_b}{\beta_F}\right]^{1/2} \tag{6.47a}$$

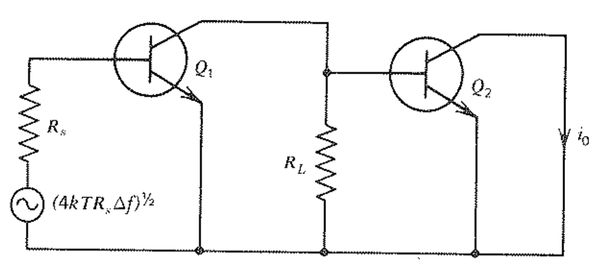


Figure 6.15. Circuit arrangement for a two-stage transistor amplifier.

Usually the effect of the second stage is quite small and no great error is made in omitting the term altogether.

We have here omitted the finite output resistance $R_0(1/R_0 = \partial I_C/\partial V_{CE})$ of the transistor. This is allowed if $R_0 \gg R_L$; otherwise a simple correction must be made.

6.3c Emitter-follower Circuit

Figure 6.16 shows an emitter-follower circuit. It has a voltage gain of about unity but a reasonably large power gain. In the figure the bias circuits are omitted.

To calculate the noise figure of a feedback circuit like the emitter follower, we short-circuit the output and count the noise of R_L as belonging to the next stage. The short-circuiting is allowed, for the ratio

$$F = \frac{\text{Voltage noise squared}}{\text{Part due to } R_s} = \frac{\text{current noise squared}}{\text{part due to } R_s}$$

If we now refer the drain noise back to the input as in Fig. 6.14*b*, we see that the equivalent circuits are the same. Hence the noise figures are also the same.

The incorporation of the noise of the load resistance R_L into the noise figure goes in the same way as in the common emitter circuit and the result is the same.

At high frequencies R_{be} is shunted by the capacitance C_{be} of the transistor. A calculation shows that this results in a large decrease in the power gain G with increasing frequency. As a consequence the circuit is no longer useful.

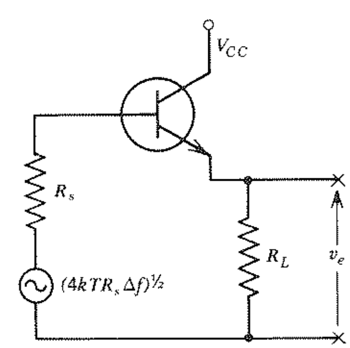


Figure 6.16. Circuit arrangement for the emitter follower.

6.3d Noise in the Common-base Circuit

It is easily seen that the common base and the common emitter circuit have identical noise figures. To that end the output is open-circuited and the equivalent circuit is slightly modified. The modification consists in describing the signal-transfer properties by a current-amplification factor α and a current generator αi_e, where i_e is the current flowing through the emitter junction impedance Z_e; in addition the impedance Z_c of the collector junction is introduced (Fig. 6.17).

The difference between the common emitter and the common base circuits is that the first is grounded at A and the second at B. As long as the noise developed across Z_c is large in comparison with the noise developed in other parts of the circuit, there is no difference in noise figures. Hence the noise figure is again given by (6.40a) and the discussion of Section 6.3a can be taken over directly.

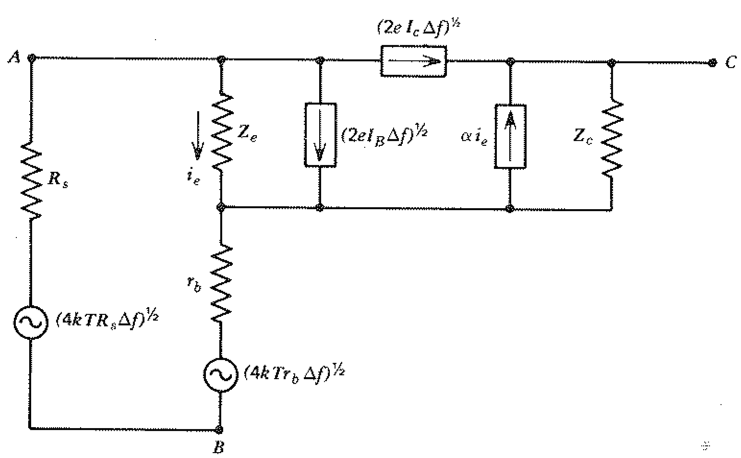


Figure 6.17. Equivalent circuit of the common-base transistor amplifier. For the common-base circuit the circuit is grounded at B, and for the common emitter circuit the circuit is grounded at A, but neither grounding alters the noise figure F.

6.3e Noise at High Frequencies

At high frequencies the capacitance C_{be} shunts R_{be}; this impairs the signal transfer from input to output and hence increases the noise figure. In that

case one obtains in reasonable approximation*

$$F_{\min}=1+x+(2x+x^2)^{1/2}, \quad \text{where} \quad x=g_m r_b\left(\frac{1}{\beta_F}+\frac{f^2}{f_T^2}\right) \tag{6.48}$$

and f_T is the frequency at which $|\beta|=1$.

6.3f The Darlington Circuit

The transistor is essentially a low-impedance device. However by using a two-stage or three-stage emitter follower one can make a circuit with a high input impedance. The reason is that the emitter-follower circuit steps up the load resistance R_E in the emitter lead by a factor $1+\beta_F$. Hence in the two-stage emitter follower of Fig. 6.18 the input impedance is

$$Z_{in}=(1+\beta_{F1})(1+\beta_{F2})R_E \tag{6.49}$$

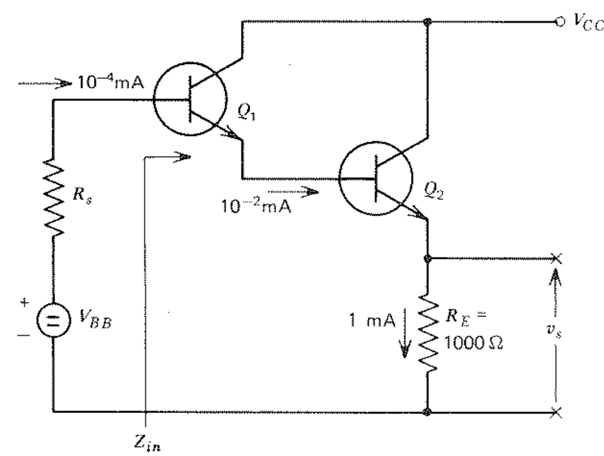


Figure 6.18. Circuit arrangement for the Darlington circuit.

and it has a voltage gain of unity. This circuit is known as the *Darlington circuit*. For $\beta_{F1}=\beta_{F2}=100$ and $R_E=1000\ \Omega$, $Z_{in}\cong 10\ \text{M}\Omega$. In comparison the input resistance of the transistor Q_1 itself is only $r_{be1}=(kT/eI_{B1})=$ 260,000 Ω for $I_{E2}=1$ mA.

*See for example, A. van der Ziel, J. A. Cruz-Emeric, R. D. Livingstone, J. C. Malpass, and D. A. McNamara, *Solid State Electronics*, **19**, 149 (1976), for a discussion and further references. A somewhat more accurate expression is obtained by replacing $g_m r_b$ by $G_1(f)+g_m r_b$, where the function $G_1(f)$ is tabulated by the authors and defined as $G_1(f)=(R_e+R_{cor})/R_{e0}$. Here R_e is the high-frequency emitter resistance, $R_{e0}=kT/eI_E$ is its low-frequency value, and the correlation resistance, R_{cor}, is defined in the paper.

If we use a high source impedance R_s, we may thus simplify the noise calculation since in that case $R_s \gg r_{be1}$ and $R_s \gg r_{b1}$. Furthermore, most of the noise comes from the first stage; the noise of the second stage is negligible. Equation (6.40a) then becomes

$$F = 1 + \frac{1}{2}\frac{R_s}{r_{be1}} + \frac{1}{2g_m R_s}\left(\frac{R_s}{r_{be1}}\right)^2 = 1 + \frac{1}{2}\frac{R_s}{r_{be1}}(1 + 1/\beta_F) \cong 1 + \frac{1}{2}\frac{R_s}{r_{be1}} \quad (6.50)$$

For $I_{E2} = 1$ mA, $r_{be1} = 260{,}000\ \Omega$; hence for $R_s = 10^7\ \Omega$, $F \cong 20$. The Darlington circuit thus has a large noise figure when operated from a high source impedance. This should be taken into account in applications.

7

FLICKER NOISE AND GENERATION–RECOMBINATION NOISE

Although the exact cause of flicker noise is not known for *all* devices showing it, it is usually assumed that flicker noise in semiconductor resistors, MOSFETs, and transistors is due to generation–recombination noise with a distribution of time constants. Such a distribution is attributed to the interaction of current carriers with traps in the surface oxide. We discuss this explanation in Section 7.1, in Section 7.2 we apply it to MOSFETs, and in Section 7.3, to transistors. In Section 7.4 we discuss flicker noise in carbon resistors.

In Section 7.5 we discuss generation–recombination noise in JFETs. We shall see that there are two causes for it: (a) noise due to generation–recombination centers in the space-charge region of the junctions (the predominant effect at or near room temperature), and (b) noise due to traps or impurity centers in the conducting channel; this effect predominates at low temperatures.

7.1 DERIVATION OF FLICKER NOISE FORMULAS FROM GENERATION–RECOMBINATION NOISE

If a voltage V is applied to a semiconductor sample in which the number N of carriers fluctuates, and the d.c. current flowing through the sample is I_0, then the spectral intensity of the fluctuating current is given by (5.51a) as

$$S_I(f) = 4\frac{I_0^2}{N_0^2}\overline{\Delta N^2}\frac{\tau}{1 + \omega^2\tau^2} \quad (7.1)$$

Here N_0 is the equilibrium number of carriers, $\Delta N = (N - N_0)$, $\overline{\Delta N^2}$ its